Multivariable (φ, Γ)-modules and representations of products of Galois groups

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Abstract

For a prime number p, let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$ and residue field κ_L. We also let n be a positive integer. In this thesis we describe the category of finitely generated continuous representations of the n-th direct power of the absolute Galois group G_L of L with coefficients in $\mathcal{O}_L$ using a generalized version of Fontaine's (φ, Γ)-modules.

In Chapter 4 we prove that the category of continuous representations of the n-th direct power of G_L on finite dimensional κ_L-vector spaces is equivalent to the category of étale (φ, Γ)-modules over a n-variable Laurent series ring over κ_L. In Chapter 5 we extend this equivalence to prove that the category of continuous representations of the n-th direct power of G_L on finitely generated $\mathcal{O}_L$-modules is equivalent to the category of étale (φ, Γ)-modules over a n-variable Laurent series ring over $\mathcal{O}_L$.

On the one hand, if we let $n = 1$ and L be arbitrary, we obtain the refinement of Fontaine's original construction due to Kisin, Rin and Schneider, which uses Lubin-Tate theory. On the other hand, if we let n be arbitrary and $L = \mathbb{Q}_p$, we recover Zábrádi's theory of multivariable cyclotomic (φ, Γ)-modules that generalizes Fontaine's use of a single free variable. Therefore, our thesis provides a common framework for both of these generalizations.

Zusammenfassung

Für eine Primzahl p, sei L eine endliche Erweiterung von $\mathbb{Q}_p$ mit Ganzheitsring $\mathcal{O}_L$ und Restklassenkörper κ_L. Sei ferner n eine positive ganze Zahl. In dieser Arbeit beschreiben wir die Kategorie der endlich erzeugten stetigen Darstellungen der n-ten direkten Potenz der absoluten Galoisgruppe G_L von L mit Koeffizienten in $\mathcal{O}_L$, unter Verwendung einer verallgemeinerten Version der (φ, Γ)-Moduln von Fontaine.

In Kapitel 4 beweisen wir, dass die Kategorie der stetigen Darstellungen der n-ten direkten Potenz von G_L auf endlichen dimensionalen κ_L-Vektorräumen und die Kategorie étaler (φ, Γ)-Moduln über einem n-variablen Laurentreihenring über κ_L äquivalent sind. In Kapitel 5 erweitern wir diese Äquivalenz, um zu beweisen, dass die Kategorie der stetigen Darstellungen der n-ten direkten Potenz von G_L auf endlich erzeugten $\mathcal{O}_L$-Moduln und die Kategorie étaler (φ, Γ)-Moduln über einem n-variablen Laurentreihenring über $\mathcal{O}_L$ äquivalent sind.

Einerseits erhalten wir, wenn wir $n = 1$ und L willkürlich lassen, die Verfeinerung von Fontaine ursprünglicher Konstruktion gemäß Kisin, Rin und Schneider, die Lubin-Tate Theorie verwenden. Wenn wir andererseits n willkürlich lassen und $L = \mathbb{Q}_p$, erhalten wir die Theorie von Zábrádi von multivariablen zyklotomischen (φ, Γ)-Moduln, die Fontaines Verwendung einer einzelnen freien Variablen verallgemeinert. Daher bietet unsere Arbeit einen gemeinsamen Rahmen für diese beiden Verallgemeinerungen.

Contents

Introduction

Among the most central objects of attention in modern algebraic number theory and arithmetic geometry are the absolute Galois groups of local and global fields. Focusing on local fields of characteristic 0, we fix a prime number p and a finite extension L of $\mathbb{Q}_p$ with absolute Galois group $G_L := \mathrm{Gal}(\overline{L}/L)$. It is a general technique to study G_L via its representations and one obtains a rich theory when considering the category $\mathrm{Rep}_{\mathcal{O}_L}(G_L)$ of continuous representations of G_L on finitely generated modules over the ring of integers $\mathcal{O}_L$ of L.

To study such representations, Fontaine [Fon90] developed a theory of (φ, Γ)-modules, which are certain module theoretic structures that we now explain, though we do so in the more generalized framework of Kisin and Ren [KR09] that uses Lubin-Tate theory. We fix a prime element π of $\mathcal{O}_L$ and a Frobenius power series $\phi \in \mathcal{O}_L[\![X]\!]$ for π. Let $F_\phi(X,Y) \in \mathcal{O}_L[\![X,Y]\!]$ be the associated Lubin-Tate formal group law to ϕ. Adjoining to L the torsion points of $F_\phi(X,Y)$ gives an infinite Galois extension L_∞/L and we let $\Gamma_L := \mathrm{Gal}(L_\infty/L)$. The ring $\mathcal{O}_L$ embeds into the endomorphism ring $\mathrm{End}(F_\phi)$ by sending $a \in \mathcal{O}_L$ to a power series $[a]_\phi(X) \in aX + X^2\mathcal{O}_L[\![X]\!]$ in such a way that $[\pi]_\phi(X) = \phi$. Using Lubin-Tate theory, we also obtain an isomorphism $\chi_L : \Gamma_L \xrightarrow{\simeq} \mathcal{O}_L^\times$ generalizing the p-adic cyclotomic character. In Chapter 1 we provide an overview of the facts mentioned above.

Consider the ring of infinite Laurent series

$$\mathcal{A}_L := \left\{ \sum_{i \in \mathbb{Z}} a_i X^i \mid a_i \in \mathcal{O}_L, \lim_{i \to -\infty} a_i = 0 \right\}$$

which alternatively can be described as the π-adic completion of the ring of Laurent series $\mathcal{O}_L(\!(X)\!)$. The $\mathcal{O}_L$-algebra $\mathcal{A}_L$ is equipped with an $\mathcal{O}_L$-linear Frobenius endomorphism φ_L sending X to $[\pi]_\phi(X)$, and an $\mathcal{O}_L$-linear Γ_L-action

$$\begin{aligned} \Gamma_L \times \mathcal{A}_L &\longrightarrow \mathcal{A}_L \\ (\sigma, f(X)) &\longmapsto f([\chi_L(\sigma)]_\phi(X)). \end{aligned}$$

A (φ_L, Γ_L)-module over $\mathcal{A}_L$ is a finitely generated $\mathcal{A}_L$-module D which is equipped with a semilinear action of the group Γ_L and a φ_L-linear endomorphism φ_D which commutes with the Γ_L-action. Such a (φ_L, Γ_L)-module D is called *étale* if the linearized map

$$\begin{aligned}\varphi_D^{\mathrm{lin}} : \mathcal{A}_L \otimes_{\varphi_L, \mathcal{A}_L} D &\longrightarrow D \\ a \otimes d &\longmapsto a\varphi_D(d)\end{aligned}$$

is an isomorphism. Let $\mathrm{Mod}^{\text{ét}}(\varphi_L, \Gamma_L, \mathcal{A}_L)$ denote the category of étale (φ_L, Γ_L)-modules over $\mathcal{A}_L$.

Consider the case when $L = \mathbb{Q}_p, \pi = p$ and $\phi = (1+X)^p - 1$. Then F_ϕ equals the multiplicative formal group law $\widehat{\mathbb{G}}_m(X,Y) = X + Y + XY$. The field L_∞ in this case equals the field extension $\mathbb{Q}_p(\mu_{p^\infty})$ of $\mathbb{Q}_p$ obtained by adjoining the p^ℓ-th roots of unity for every $\ell \in \mathbb{N}_{\geq 1}$. Here the endomorphism $[a]_\phi(X)$ of $\widehat{\mathbb{G}}_m$ equals $(1+X)^a - 1$ for $a \in \mathbb{Z}_p$ and the isomorphism χ_L becomes the isomorphism $\Gamma_{\mathbb{Q}_p} \simeq \mathbb{Z}_p^\times$ obtained from the p-adic cyclotomic character. Fontaine [Fon90] obtained a functorial equivalence between the category of continuous finitely generated representations of $G_{\mathbb{Q}_p}$ with coefficients in $\mathbb{Z}_p$ and the category of étale $\left(\varphi_{\mathbb{Q}_p}, \Gamma_{\mathbb{Q}_p}\right)$-modules over $\mathcal{A}_{\mathbb{Q}_p}$.

Later, Kisin and Ren [KR09] generalized the result of Fontaine to an equivalence between $\mathrm{Rep}_{\mathcal{O}_L}(G_L)$ and $\mathrm{Mod}^{\text{ét}}(\varphi_L, \Gamma_L, \mathcal{A}_L)$ for arbitrary L and ϕ. However, their proof relies on results for which ϕ is constrained to be a polynomial in $\mathcal{O}_L[X]$. Schneider [Sch17] makes a further refinement and deals with the case of general Frobenius power series instead. Furthermore, instead of the original approach involving the field of norms, he uses the theory of tilting correspondence developed by Scholze in [Sch12].

The basic idea of employing (φ, Γ)-modules is expected to be indispensable for the current search of various p-adic Langlands type correspondences. Fontaine's theory of (φ, Γ)-modules was one of the starting points in the mod p and p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, which thanks to various works is now well understood (see [Bre10] for an overview).

On the other hand, in order to proceed further it also seems that Fontaine's original construction needs other kinds of modifications. One such modification would be to develop a *multivariable* theory of (φ, Γ)-modules. There are various concepts involving (φ, Γ)-modules over coefficient rings of several free variables. Berger [Ber13] introduces a theory of (φ, Γ)-modules over coefficient rings whose number of free variables depends on the extension $L/\mathbb{Q}_p$. These (φ, Γ)-modules seem to be suitable for the p-adic Langlands program for GL_2 over a finite nontrivial extension of $\mathbb{Q}_p$.

We focus instead on Zábrádi's concept of multivariable (φ, Γ)-modules [Záb18a]. Motivated by an attempt to generalize the p-adic Langlands program to $\mathrm{GL}_n(\mathbb{Q}_p)$, he introduces (φ, Γ)-modules over the rings $A[\![X_1, \ldots, X_n]\!][\prod_{1 \leq i \leq n} X_i^{-1}]$ where A is a finite quotient of the ring of integers in a finite extension of $\mathbb{Q}_p$ and X_i are independent commuting free variables. For each $1 \leq i \leq n$ we have an action of $\Gamma_{\mathbb{Q}_p} \simeq \mathbb{Z}_p^\times$ given by inserting X_i into the formal group law $\widehat{\mathbb{G}}_m$ and a Frobenius operator φ_i sending X_i to $(1+X_i)^p - 1$ while leaving the other variables unchanged. In [Záb18b] he uses these n-variable (φ, Γ)-modules to classify p-adic and p-torsion representations of the n-fold product of $G_{\mathbb{Q}_p}$. Another motivation for the relevance of such representations is explained in the last paragraph of the introduction of [Záb18b]. There, Zábrádi explains the connection between his multivariable (φ, Γ)-modules and a form of Drinfeld's lemma for perfectoid spaces. This is also discussed in Section 1.3 of [CKZ21].

In our thesis, we generalize Zábrádi's multivariable cyclotomic (φ, Γ)-modules and extend the results of [Záb18b] by replacing $\mathbb{Q}_p$ with our local field L. In doing so, we will use the framework of Kisin, Ren and Schneider by allowing general Lubin-Tate formal group laws instead of $\widehat{\mathbb{G}}_m$.

In Chapter 2 we define the relevant categories. Let n be a positive integer and Δ_n denote the set $\{1, \ldots, n\}$. We fix n Frobenius power series $\phi_1, \ldots, \phi_n$ for our π. We also consider the n-fold products $G_{\Delta_n,L} = \prod_{i \in \Delta_n} G_L$ and $\Gamma_{\Delta_n,L} = \prod_{i \in \Delta_n} \Gamma_L$. On the one hand, we have the category $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ of continuous finitely generated representations of $G_{\Delta_n,L}$ with coefficients in $\mathcal{O}_L$.

For the coefficient ring of the other category, we let $\mathcal{A}_{\Delta_n}$ be the π-adic completion of the $\mathcal{O}_L$-algebra $\mathcal{O}_L[\![X_1, \ldots, X_n]\!][(X_1 \ldots X_n)^{-1}]$. On $\mathcal{A}_{\Delta_n}$, for each $i \in \Delta_n$, we have an $\mathcal{O}_L$-linear Frobenius operator φ_i mapping X_i to $[\pi]_{\phi_i}(X_i)$ and leaving the other variables unchanged. We also have an $\mathcal{O}_L$-linear action of $\Gamma_{\Delta_n,L}$ on $\mathcal{A}_{\Delta_n}$ by letting an element $\prod_{i \in \Delta_n} \sigma_i \in \Gamma_{\Delta_n,L}$ map X_i to $[\chi_L(\sigma_i)]_{\phi_i}(X_i)$ for $i \in \Delta_n$.

Definition 0.1. *A $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-module over $\mathcal{A}_{\Delta_n}$ is a finitely generated $\mathcal{A}_{\Delta_n}$-module D together with commuting semilinear actions of the operators φ_i and group $\Gamma_{\Delta_n,L}$. We say that D is étale if the map*

$$\begin{aligned} \varphi_i^{\mathrm{lin}} : \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D &\longrightarrow D \\ a \otimes d &\longmapsto a\varphi_i(d) \end{aligned}$$

is an isomorphism for all $i \in \Delta_n$. Let $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ denote the category of étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over $\mathcal{A}_{\Delta_n}$.

Let $E_{\Delta_n} := \mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n} \simeq \kappa_L[\![X_1, \ldots, X_n]\!][(X_1 \ldots X_n)^{-1}]$. The fact that E_{Δ_n} is no longer a field if $n > 1$ causes numerous technical problems when trying to generalize classical methods in our multivariable context. Lemma 2.1 of [Záb18a] is an important step in overcoming some of these issues in the cyclotomic setting when $L = \mathbb{Q}_p$. In [GK19], Große-Klönne states the following generalization for arbitrary L, which we prove in this thesis using a generalization of the argument used in Lemma 2.1 of [Záb18a].

Proposition 0.2 (Proposition 2.9)**.** *The only non-zero $\Gamma_{\Delta_n,L}$-invariant ideal of E_{Δ_n} is E_{Δ_n}.*

In Section 2.2 we list some immediate consequences of Proposition 0.2, the most notable one being the fact that a $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-module over E_{Δ_n} is a stably-free

E_{Δ_n}-module. This will later allow us to prove certain statements about $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over E_{Δ_n} by reducing them to the case when the underlying module is free over E_{Δ_n}.

We also note that compared to $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, we did not impose any continuity condition on the objects of $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. In Section 2.3, we define two topologies on a finitely generated $\mathcal{A}_{\Delta_n}$-module, namely the *adic topology* and the *weak topology*. In Section 2.5 we prove the following.

Theorem 0.3 (Theorem 2.42)**.** *For $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$, the action of $\Gamma_{\Delta_n,L}$ on D is continuous when D is equipped with the weak topology.*

To prove Theorem 0.3, we follow a strategy similar to that of Schneider, who proves our claim in the one variable case, see Theorem 2.2.8 in [Sch17]. For our first step, we reduce our statement to the case when D is annihilated by a power of π. Then we look for a basic system of open neighbourhoods of zero inside D that covers D and is preserved under the action of $\Gamma_{\Delta_n,L}$. We construct such open neighbourhoods using a lattice D^{++} inside D. Our definition of D^{++} uses the adic topology of D and it recovers the one in the one-variable case due to Colmez [Col10]. Due to the more complicated nature of $\mathcal{A}_{\Delta_n}$, which is no longer a PID when $n > 1$, we modify the remaining steps of Schneider's proof in the one variable case. Our modifications can also be applied in the one variable case to shorten some of the arguments in [Sch17].

Before we describe the functors between the two categories, in Section 3.1 we define a rather large ring $\mathbb{C}^\flat_{p,\Delta_n}$ of characteristic p, which is a multivariable version of $\mathbb{C}^\flat_p$, where the latter denotes the tilt of the completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}_p}$. We will also see in Section 3.1 that the ring $\mathbb{C}^\flat_{p,\Delta_n}$ is a perfect κ_L-algebra which has a natural topology and a $G_{\Delta_n,L}$-action. In Section 3.2 we explain how the relevant rings of characteristic p embed into $\mathbb{C}^\flat_{p,\Delta_n}$. In Section 3.3 we consider the ring of ramified Witt vectors $W(\mathbb{C}^\flat_{p,\Delta_n})_L$. This ring will embed the relevant rings of characteristic 0. The topology and the Galois action on $\mathbb{C}^\flat_{p,\Delta_n}$ are naturally extended to $W(\mathbb{C}^\flat_{p,\Delta_n})_L$.

We let $\mathcal{A}_L^{\mathrm{ur}}$ be the ring of integers of the maximal unramified extension of $\mathrm{Frac}(\mathcal{A}_L)$. Consider the n-fold products $\mathcal{A}_{\Delta_n,\circ} = \bigotimes_{i\in\Delta_n,\mathcal{O}_L} \mathcal{A}_L$ and $\mathcal{A}^{\mathrm{ur}}_{\Delta_n,\circ} = \bigotimes_{i\in\Delta_n,\mathcal{O}_L} \mathcal{A}^{\mathrm{ur}}_L$, as well as the $\mathcal{O}_L$-algebra $\mathcal{A}^{\mathrm{ur}}_{\Delta_n} = \mathcal{A}^{\mathrm{ur}}_{\Delta_n,\circ} \otimes_{\mathcal{A}_{\Delta_n,\circ}} \mathcal{A}_{\Delta_n}$. We let $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ denote the π-adic completion of $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$. Using that φ_L can be extended to a Frobenius operator on $\mathcal{A}^{\mathrm{ur}}_L$, for every $i \in \Delta_n$ we will be able to extend the existing operators φ_i on $\mathcal{A}_{\Delta_n}$ to Frobenius operators on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$, which we will denote by φ_i as well. To define a topology and a $G_{\Delta_n,L}$-action on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$, we embed the latter into $W(\mathbb{C}^\flat_{p,\Delta_n})_L$. To do so, in Section 3.4 we first embed $\mathcal{A}_{\Delta_n}$ into $W(\mathbb{C}^\flat_{p,\Delta_n})_L$. This embedding will be compatible with the Galois action and the weak topology on $\mathcal{A}_{\Delta_n}$. This embedding will also induce the π-adic topology on $\mathcal{O}_L$. In Section 3.5 we extend this embedding to $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$. In this section we further prove that all of our topological embeddings are maintained when we take quotients by a power of π in our rings.

For our first functor, given $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$, the $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-module $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}}$

D carries a diagonal $G_{\Delta_n,L}$-action where $G_{\Delta_n,L}$ acts on D through its quotient $\Gamma_{\Delta_n,L}$. The space $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D$ also carries diagonal Frobenius operators φ_i using the existing ones on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ and D. We define

$$\mathbb{T}(D) \coloneqq \bigcap_{i\in\Delta_n} \left(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D\right)^{\varphi_i=\mathrm{id}}.$$

Proving that the action of $G_{\Delta_n,L}$ on $\mathbb{T}(D)$ is continuous is rather subtle and motivates the majority of our constructions in Chapters 2 and 3.

For a functor in the opposite direction, given $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, consider the $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-module $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T$ which carries a diagonal $G_{\Delta_n,L}$-action and the Frobenius endomorphisms $\varphi_i \coloneqq \varphi_i \otimes \mathrm{id}$. We define

$$\mathbb{D}(T) \coloneqq \left(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T\right)^{H_{\Delta_n,L}}.$$

In the next chapters we show that $\mathbb{D}$ and $\mathbb{T}$ give quasi-inverse functors between $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ which realize the equivalence between these two categories.

In Chapter 4, we focus our attention on the objects of our categories that are annihilated by π. We let $\mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$ denote the category of continuous representations of $G_{\Delta_n,L}$ on finite dimensional κ_L-vector spaces and we let $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ denote the category of étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over E_{Δ_n}. The main theorem of this chapter is the following.

Theorem 0.4 (Theorem 4.36)**.** *The functors $\mathbb{D}$ and $\mathbb{T}$ give an equivalence of categories between $\mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$.*

The main reference for Chapter 4 is [Záb18b] and most of the proofs here are straightforward adaptations of their counterparts in the aforementioned paper. Some proofs were restructured and provided additional details. The fact that we proved Proposition 0.2 is an important reason that allows us to use Zábrádi's line of arguments in this case. The main idea of his inductive proof is to build a lattice $D^{+*}_{\overline{n}}$ using a multivariable analog of the Colmez module D^+. From this lattice one can obtain multivariable (φ, Γ)-modules over $n-1$ variables that are used in the inductive step.

In Chapter 5 we prove the main theorem of our thesis.

Theorem 0.5 (Theorem 5.23)**.** *The functors $\mathbb{D}$ and $\mathbb{T}$ give an equivalence of categories between $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$.*

The strategy will be to use Theorem 0.4 as a starting point. After establishing some exactness properties of $\mathbb{D}$ and $\mathbb{T}$, a dévissage argument shows that for $m \geq 1$, the étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over $\mathcal{A}_{\Delta_n}$ annihilated by π^m classify the continuous representations of $G_{\Delta_n,L}$ with coefficients in $\mathcal{O}_L/\pi^m\mathcal{O}_L$. For our final step, we use some projective limit arguments to complete the proof our main theorem in Section 5.2. We will also extend our equivalence over other coefficients in Section 5.3.

We also believe that our theory can be generalized without major modifications to handle p-adic continuous representations of products $G_{L_1} \times \ldots \times G_{L_n}$ of absolute Galois groups of possibly distinct finite extension of $\mathbb{Q}_p$, which recovers our results when taking $L_1 = \ldots = L_n = L$.

Chapter 1

Preliminaries

In this chapter we provide the necessary background and establish some of the notation that we use throughout the text. Let p be a prime number and L be a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. We denote by $\mathcal{O}_L$ the ring of integers in L and by κ_L its residue field. Suppose that $|\kappa_L| = q = p^{\mathfrak{f}}$ for some $\mathfrak{f} \in \mathbb{N}_{\geq 1}$. We also fix a prime element π of $\mathcal{O}_L$.

In Section 1.1 we provide a brief overview of the theory of ramified Witt vectors. This is followed by our treatment of Lubin-Tate formal group laws and Lubin-Tate extensions in Section 1.2. In Section 1.3 we associate to the completion of an algebraic closure of L a field $\mathbb{C}_p^\flat$ of characteristic p using the tilting construction from [Sch12]. We explain here the main properties of $\mathbb{C}_p^\flat$ and the underlying action of the absolute Galois group of L on $\mathbb{C}_p^\flat$. In Section 1.4 we explain how one can naturally assign a topology on a finitely generated module over a given topological ring. Finally, in Section 1.5 we collect some results from commutative algebra that we will use in the later parts of the thesis.

1.1 Ramified Witt vectors

In this section we recall the basics of ramified Witt vectors for which we follow the treatment of Section 1.1 of [Sch17].

Definition 1.1. *For any integer $m \geq 0$ we call*

$$\Phi_m(X_0, \ldots, X_m) \coloneqq X_0^{q^m} + \pi X_1^{q^{m-1}} + \ldots + \pi^m X_m$$

the m-th Witt polynomial.

Let B be a commutative $\mathcal{O}_L$-algebra and let

$$B^{\mathbb{N}_0} := \{(b_0, b_1, \ldots) : b_j \in B\}$$

be the countably infinite direct product of the algebra B with itself, in which addition and multiplication are componentwise. We introduce the maps

$$\begin{aligned} f_B : B^{\mathbb{N}_0} &\longrightarrow B^{\mathbb{N}_0} \\ (b_0, b_1, \ldots) &\longmapsto (b_1, b_2, \ldots), \\ \Phi_m : B^{\mathbb{N}_0} &\longrightarrow B \\ (b_0, b_1, \ldots) &\longmapsto \Phi_m(b_0, \ldots, b_m), \end{aligned}$$

for $m \geq 0$, and

$$\begin{aligned} \Phi_B : B^{\mathbb{N}_0} &\longrightarrow B^{\mathbb{N}_0} \\ \mathbf{b} &\longmapsto (\Phi_0(\mathbf{b}), \Phi_1(\mathbf{b}), \Phi_2(\mathbf{b}) \ldots). \end{aligned}$$

Lemma 1.2. *Let B be a commutative $\mathcal{O}_L$-algebra and $\mathbf{b} = (b_j)_j$, $\mathbf{u} = (u_j)_j \in B^{\mathbb{N}_0}$ such that $\Phi_B(\mathbf{b}) = \mathbf{u}$. Let C be an $\mathcal{O}_L$-subalgebra of B with the property that multiplication by π is injective on B/C. Then for every $m \geq 0$ we have that $u_0, \ldots, u_m \in C$ if and only if $b_0, \ldots, b_m \in C$.*

Proof. See Remark 1.1.5 in [Sch17]. □

Consider the polynomial $\mathcal{O}_L$-algebra

$$A := \mathcal{O}_L[X_0, X_1, \ldots, Y_0, Y_1, \ldots]$$

in two sets of countably many variables. Let $\mathbf{X} := (X_0, X_1, \ldots)$ and $\mathbf{Y} := (Y_0, Y_1, \ldots)$ in $A^{\mathbb{N}_0}$. It is proven in Section 1.1 of [Sch17] that there exist uniquely determined elements $\mathbf{S} = (S_j)_j$, $\mathbf{P} = (P_j)_j$, $\mathbf{I} = (I_j)_j$ and $\mathbf{F} = (F_j)_j$ in $A^{\mathbb{N}_0}$ such that

$$\begin{aligned} \Phi_A(\mathbf{S}) &= \Phi_A(\mathbf{X}) + \Phi_A(\mathbf{Y}), \\ \Phi_A(\mathbf{P}) &= \Phi_A(\mathbf{X})\Phi_A(\mathbf{Y}), \\ \Phi_A(\mathbf{I}) &= -\Phi_A(\mathbf{X}), \\ \Phi_A(\mathbf{F}) &= f_A(\Phi_A(\mathbf{X})), \end{aligned}$$

respectively. This means that we have

$$\begin{aligned} \Phi_m(S_0, \ldots, S_m) &= \Phi_m(X_0, \ldots, X_m) + \Phi_m(Y_0, \ldots, Y_m), \\ \Phi_m(P_0, \ldots, P_m) &= \Phi_m(X_0, \ldots, X_m)\Phi_m(Y_0, \ldots, Y_m), \\ \Phi_m(I_0, \ldots, I_m) &= -\Phi_m(X_0, \ldots, X_m), \\ \Phi_m(F_0, \ldots, F_m) &= \Phi_{m+1}(X_0, \ldots, X_{m+1}) \end{aligned} \tag{1.1}$$

for every $m \geq 0$. Lemma 1.2 implies that

$$\begin{aligned} S_m, P_m &\in \mathcal{O}_L[X_0, \ldots, X_m, Y_0, \ldots, Y_m], \\ I_m &\in \mathcal{O}_L[X_0, \ldots, X_m], \\ F_m &\in \mathcal{O}_L[X_0, \ldots, X_{m+1}]. \end{aligned}$$

In Section 1.1 of [Sch17] it is also proven that for every $a \in \mathcal{O}_L$, there exist uniquely determined elements $\Omega_j(a) \in \mathcal{O}_L$ for $j \geq 0$ such that

$$\Phi_{\mathcal{O}_L}(\Omega(a)) = (a, a, \ldots)$$

where $\Omega(a) := (\Omega_j(a))_j$.

Example 1.3. The elements S_m, P_m, I_m can be computed recursively from the system of equations (1.1). Since $\Phi_0(S_0) = \Phi_0(X_0) + \Phi_0(Y_0)$ and $\Phi_0(P_0) = \Phi_0(X_0)\Phi_0(Y_0)$, we have that $S_0 = X_0 + Y_0$ and $P_0 = X_0 Y_0$. Similarly, from $\Phi_0(I_0) = -\Phi_0(X_0)$, we have that $I_0 = -X_0$. When $p > 2$, we can inductively show that $I_m = -X_m$ for all $m \geq 0$.

It is more difficult to write S_m and P_m explicitly for arbitrary values of m. Similarly, it is also more difficult to write I_m explicitly for arbitrary values of m when $p = 2$. Nevertheless, it is easy to prove that their degrees are bounded by a function on m.

Lemma 1.4. *(i) The degree of $S_m(X_0, \ldots, X_m, Y_0, \ldots, Y_m)$ is at most q^m.*

(ii) The degree of $I_m(X_0, \ldots, X_m)$ is at most q^m.

(iii) The degree of $P_m(X_0, \ldots, X_m, Y_0, \ldots, Y_m)$ is at most $2q^m$.

Proof. (i) We show this by induction on $m \geq 0$. For $m = 0$, $S_0 = X_0 + Y_0$ and the claim follows. Let $m \geq 1$ and assume that the claim holds for all $0 \leq j \leq m - 1$. Using the defining identity

$$\Phi_m(S_0, \ldots, S_m) = \Phi_m(X_0, \ldots, X_m) + \Phi_m(Y_0, \ldots, Y_m)$$

we obtain that

$$\pi^m S_m = \Phi_m(X_0, \ldots, X_m) + \Phi_m(Y_0, \ldots, Y_m) - \left(S_0^{q^m} + \pi S_1^{q^{m-1}} + \ldots + \pi^{m-1} S_{m-1}^q\right).$$

The degrees of $\Phi_m(X_0, \ldots, X_m)$ and $\Phi_m(Y_0, \ldots, Y_m)$ are q^m and the degree of $S_j^{q^{m-j}}$ is at most q^m for every $0 \leq j \leq m - 1$ by the induction hypothesis.

(ii) This is also proven by induction on $m \geq 0$. For $m = 0$, $I_0 = -X_0$ and the claim follows. Let $m \geq 1$ and assume that the claim holds for all $0 \leq j \leq m - 1$. Using the defining identity

$$\Phi_m(I_0, \ldots, I_m) = -\Phi_m(X_0, \ldots, X_m),$$

we obtain that

$$\pi^m I_m = -\Phi_m(X_0, \ldots, X_m) - \left(I_0^{q^m} + \pi I_1^{q^{m-1}} + \ldots + \pi^{m-1} I_{m-1}^q\right).$$

The degree of $\Phi_m(X_0, \ldots, X_m)$ is q^m and the degree of $I_j^{q^{m-j}}$ is at most q^m for every $0 \leq j \leq m - 1$ by the induction hypothesis.

(iii) We induct on $m \geq 0$ again. For $m = 0$ the claim is clear because $P_0 = X_0 Y_0$. Let $m \geq 1$ and assume that the claim holds for all $0 \leq j \leq m-1$. Using the defining identity

$$\Phi_m(P_0, \ldots, P_m) = \Phi_m(X_0, \ldots, X_m)\Phi_m(Y_0, \ldots, Y_m),$$

we obtain that

$$\pi^m P_m = \Phi_m(X_0, \ldots, X_m)\Phi_m(Y_0, \ldots, Y_m) - \left(P_0^{q^m} + \pi P_1^{q^{m-1}} + \ldots + \pi^{m-1} P_{m-1}^q\right).$$

The claim follows, as the degree of $\Phi_m(X_0, \ldots, X_m)\Phi_m(Y_0, \ldots, Y_m)$ is $2q^m$ and the degree of $P_j^{q^{m-j}}$ is at most $2q^j q^{m-j} = 2q^m$ for every $0 \leq j \leq m-1$ by the induction hypothesis. $\square$

Let B again be an arbitrary commutative $\mathcal{O}_L$-algebra. On the one hand we have the $\mathcal{O}_L$-algebra $(B^{\mathbb{N}_0}, +, \cdot)$ defined as a direct product. On the other hand we define on the set $W(B)_L := B^{\mathbb{N}_0}$ a new *addition*

$$(a_j)_j \boxplus (b_j)_j := (S_j(a_0, \ldots, a_j, b_0, \ldots, b_j))_j$$

and a new *multiplication*

$$(a_j)_j \boxdot (b_j)_j := (P_j(a_0, \ldots, a_j, b_0, \ldots, b_j))_j.$$

Moreover we put

$$\mathbf{0} := (0, 0, \ldots) \text{ and } \mathbf{1} := (1, 0, 0, \ldots).$$

Because of (1.1) the map

$$\Phi_B : W(B)_L \longrightarrow B^{\mathbb{N}_0}$$

satisfies the identities

$$\begin{aligned}
\Phi_B(\mathbf{a} \boxplus \mathbf{b}) &= \Phi_B(\mathbf{a}) + \Phi_B(\mathbf{b}), \\
\Phi_B(\mathbf{a} \boxdot \mathbf{b}) &= \Phi_B(\mathbf{a}) \cdot \Phi_B(\mathbf{b}), \\
\Phi_B(\mathbf{0}) &= (0, 0, \ldots), \\
\Phi_B(\mathbf{1}) &= (1, 1, \ldots).
\end{aligned}$$

Any homomorphism of commutative $\mathcal{O}_L$-algebras $\rho : B_1 \longrightarrow B_2$ induces the $\mathcal{O}_L$-algebra homomorphism

$$\begin{aligned}
\rho^{\mathbb{N}_0} : B_1^{\mathbb{N}_0} &\longrightarrow B_2^{\mathbb{N}_0} \\
(b_m)_m &\longmapsto (\rho(b_m)_m).
\end{aligned}$$

The map $W(\rho)_L := \rho^{\mathbb{N}_0} : W(B_1)_L \longrightarrow W(B_2)_L$ commutes with $\boxplus$ and $\boxdot$, satisfies $W(\rho)_L(\mathbf{1}) = \mathbf{1}$ and makes the diagram

$$\begin{tikzcd}
W(B_1)_L \arrow[r, "\Phi_{B_1}"] \arrow[d, "W(\rho)_L"'] & B_1^{\mathbb{N}_0} \arrow[d, "\rho^{\mathbb{N}_0}"] \\
W(B_2)_L \arrow[r, "\Phi_{B_2}"] & B_2^{\mathbb{N}_0}.
\end{tikzcd}$$

commute. One uses this to show the following.

Proposition 1.5. *(i)* $(W(B)_L, \boxplus, \boxdot)$ *is a commutative ring with zero element* $\mathbf{0}$ *and unit element* $\mathbf{1}$*. The additive inverse of* $(b_j)_j \in W(B)_L$ *is* $(I_j(b_0, \ldots, b_j))_j$*.*

(ii) The map $\Omega : \mathcal{O}_L \longrightarrow (W(B)_L, \boxplus, \boxdot)$ *is a ring homomorphism, making* $(W(B)_L, \boxplus, \boxdot)$ *into an* $\mathcal{O}_L$*-algebra.*

(iii) The map $\Phi_B : W(B)_L \longrightarrow B^{\mathbb{N}_0}$ *is a homomorphism of* $\mathcal{O}_L$*-algebras; in particular, for any* $m \geq 0$*,*

$$\begin{aligned} \Phi_m : W(B)_L &\longrightarrow B \\ (b_j)_j &\longmapsto \Phi_m(b_0, \ldots, b_m) \end{aligned}$$

is a homomorphism of $\mathcal{O}_L$*-algebras.*

(iv) Given a homomorphism of commutative $\mathcal{O}_L$*-algebras* $\rho : B_1 \longrightarrow B_2$*, the map*

$$W(\rho)_L : W(B_1)_L \longrightarrow W(B_2)_L$$

is an $\mathcal{O}_L$*-algebra homomorphism as well.*

Proof. See Proposition 1.1.8 in [Sch17]. □

Definition 1.6. *We call* $(W(B)_L, \boxplus, \boxdot)$ *the ring of ramified Witt vectors with coefficients in* B*.*

In addition, we have on $W(B)_L$ the maps

$$\begin{aligned} F : W(B)_L &\longrightarrow W(B)_L \\ (b_m)_m &\longmapsto (F_m(b_0, \ldots, b_{m+1}))_m \end{aligned}$$

and

$$\begin{aligned} V : W(B)_L &\longrightarrow W(B)_L \\ (b_m)_m &\longmapsto (0, b_0, b_1 \ldots), \end{aligned}$$

called the *Frobenius* and the *Verschiebung*, respectively. For any $m \in \mathbb{N}_{\geq 1}$ define

$$V_m(B)_L := \mathrm{im}(V^m) = \{(b_j)_j \in W(B)_L : b_0 = \ldots = b_{m-1} = 0\}.$$

It is proven in Proposition 1.1.10 (i) and (ii) of [Sch17] that F is an $\mathcal{O}_L$-algebra endomorphism of $W(B)_L$, while V is an $\mathcal{O}_L$-module endomorphism of $W(B)_L$. By Proposition 1.1.10 (iv) in [Sch17], we also know that $V_m(B)_L$ is an ideal of $W(B)_L$ for every $m \geq 1$. We call

$$W_m(B)_L := W(B)_L / V_m(B)_L$$

the ring of ramified Witt vectors of length m with coefficients in B.

Lemma 1.7. *Let* B *be a commutative* $\mathcal{O}_L$*-algebra and* $m \in \mathbb{N}$*.*

(i) Let $(a_j)_j$, $(b_j)_j \in W(B)_L$ *such that* a_j *or* b_j *equals* 0 *for every* $0 \leq j \leq m$. *Then*

$$(a_j)_j \boxplus (b_j)_j = (a_0 + b_0, a_1 + b_1, \ldots, a_m + b_m, \ldots).$$

(ii) The map

$$\begin{aligned} W(B)_L &\longrightarrow \varprojlim_{j\geq 1} W_j(B)_L \\ \mathbf{b} &\longmapsto (\mathbf{b} \boxplus V_j(B)_L)_{j\geq 1} \end{aligned}$$

is an isomorphism of $\mathcal{O}_L$*-algebras.*

Proof. See Lemma 1.1.13 and Exercise 1.1.14 in [Sch17]. $\square$

Lemma 1.8. *For a commutative* κ_L*-algebra* B *and any* $\mathbf{b} = (b_m)_m \in W(B)_L$ *we have that*

(i) $F(\mathbf{b}) = (b_m^q)_m$,

(ii) $\pi\mathbf{b} = (0, b_0^q, b_1^q, \ldots)$,

(iii) $\pi^m W(B)_L \subseteq V_m(B)_L$ *for all* $m \geq 1$,

(iv) The natural map

$$\begin{aligned} W(B)_L &\longrightarrow \varprojlim_{m} W(B)_L/\pi^m W(B)_L \\ \mathbf{b} &\longmapsto (\mathbf{b} \boxplus \pi^m W(B)_L)_m \end{aligned}$$

is an isomorphism.

Proof. See Proposition 1.1.18 in [Sch17]. $\square$

Proposition 1.9. *If* B *is a commutative perfect* κ_L*-algebra, then*

(i) $\pi 1_{W(B)_L} \neq 0$ *is not a zero divisor in* $W(B)_L$,

(ii) $V_m(B)_L = \pi^m W(B)_L$ *for any* $m \geq 0$.

Proof. See Proposition 1.1.19 in [Sch17]. $\square$

Proposition 1.10. *Let* B *be a commutative* $\mathcal{O}_L$*-algebra. Suppose that* $\pi 1_B$ *is not a zero divisor in* B *and that* B *has an endomorphism of* $\mathcal{O}_L$*-algebras* ψ *such that*

$$\psi(b) \equiv b^q \bmod \pi B$$

for every $b \in B$. *Then there is a unique homomorphism of* $\mathcal{O}_L$*-algebras*

$$s_B : B \longrightarrow W(B)_L$$

such that $\Phi_m \circ s_B = \psi^m$ *for any* $m \geq 0$. *The map* s_B*, equivalently, is uniquely determined by the requirements that* $\Phi_0 \circ s_B = \mathrm{id}_B$ *and* $s_B \circ \psi = F \circ s_B$.

Proof. See Lemma 1.1.23 in [Sch17]. □

Note that taking $B = \mathcal{O}_L$ and $\psi = \mathrm{id}_{\mathcal{O}_L}$ in Proposition 1.10, we obtain the map

$$s_{\mathcal{O}_L} : \mathcal{O}_L \to W(\mathcal{O}_L)_L.$$

Lemma 1.11. *The composition of the maps*

$$\mathcal{O}_L \xrightarrow{s_{\mathcal{O}_L}} W(\mathcal{O}_L)_L \xrightarrow{W(\mathrm{pr})_L} W(\kappa_L)_L$$

is an isomorphism, where $\mathrm{pr} : \mathcal{O}_L \to \kappa_L$ *is the natural projection map.*

Proof. See Corollary 1.1.25 of [Sch17]. □

1.2 Lubin-Tate formal group laws

In this section we briefly discuss Lubin Tate formal group laws and Lubin-Tate extensions, which have important applications in class field theory. For simplicity we will only consider the formal group laws with coefficients in $\mathcal{O}_L$, even though one can allow coefficients from more general rings. The main references that we follow are [Haz78], [Sch17] and [Mil20].

Definition 1.12. *A commutative formal group law over* $\mathcal{O}_L$ *is a formal power series* $F(X,Y) \in \mathcal{O}_L[\![X,Y]\!]$ *in two variables with coefficients in* $\mathcal{O}_L$ *such that*

(i) $F(X,0) = X$ *and* $F(0,Y) = Y$,

(ii) $F(X, F(Y,Z)) = F(F(X,Y),Z)$,

(iii) $F(X,Y) = F(Y,X)$.

Remark 1.13. In (A.4.7) of [Haz78] it is proven that for a commutative formal group law $F(X,Y)$ there exists a unique formal power series $\iota_F(X) \in \mathcal{O}_L[\![X]\!]$ such that

(i) $\iota_F(X) \in -X + X^2\mathcal{O}_L[\![X]\!]$ and

(ii) $F(X, \iota_F(X)) = 0$.

Example 1.14. The formal group law

$$\widehat{\mathbb{G}}_m(X,Y) \coloneqq X + Y + XY = (1+X)(1+Y) - 1$$

is called the *multiplicative formal group law*. In this case

$$\iota_{\widehat{\mathbb{G}}_m} = -\frac{X}{X+1} = \sum_{j\geq 1}(-1)^j X^j.$$

Remark 1.15. A commutative formal group law F over $\mathcal{O}_L$ gives rise to abelian groups in the following way. Let K be a complete nonarchimedean extension field of L whose absolute value extends the one of L and $\mathfrak{m}_K$ denote the maximal ideal of the ring of integers of K. For any two $x, y \in \mathfrak{m}_K$ the series $x +_F y := F(x, y)$ converges with limit in $\mathfrak{m}_K$. One easily checks that $(\mathfrak{m}_K, +_F)$ is an abelian group in which the additive inverse of an element $x \in \mathfrak{m}_K$ is given by $\iota_F(x)$.

Definition 1.16. *Let $F(X, Y)$ and $G(X, Y)$ be two commutative formal group laws over $\mathcal{O}_L$. A* homomorphism *$h : F \to G$ is a formal power series $h(X) \in X\mathcal{O}_L[\![X]\!]$ such that*

$$h(F(X, Y)) = G(h(X), h(Y)).$$

When h has an inverse, meaning that there exists a homomorphism $h' : G \to F$ such that

$$h \circ h' = X = h' \circ h,$$

h is called an isomorphism*.*

Remark 1.17. The set $\mathrm{End}(F)$ of homomorphisms from F to F is a (possibly noncommutative) ring with respect to the addition $(h_1 + h_2)(X) := F(h_1(X), h_2(X))$ and the multiplication $(h_1 \cdot h_2)(X) := h_1(h_2(X))$. The zero and unit elements of $\mathrm{End}(F)$ are 0 and X, respectively. Moreover, if K is an extension field of L that is complete with respect to an absolute value extending the one of L, then any $h \in \mathrm{End}(F)$ induces the endomorphism $x \mapsto h(x)$ of the abelian group $(\mathfrak{m}_K, +_F)$. When $F = \widehat{\mathbb{G}}_m$, the map sending x to $1+x$ defines an isomorphism between $(\mathfrak{m}_K, +_{\widehat{\mathbb{G}}_m})$ and the subgroup $1 + \mathfrak{m}_K$ of $K^\times$.

Recall that we have fixed a prime element π of $\mathcal{O}_L$.

Definition 1.18. *A Frobenius power series for π is a formal power series $\phi(X) \in \mathcal{O}_L[\![X]\!]$ such that*

(i) $\phi(X) \in \pi X + X^2\mathcal{O}_L[\![X]\!]$,

(ii) $\phi(X) \equiv X^q \bmod \pi\mathcal{O}_L[\![X]\!]$.

Example 1.19. (i) The polynomial $\phi(X) = \pi X + X^q$ is a Frobenius power series for π.

(ii) When $L = \mathbb{Q}_p$ and $\pi = p$, the polynomial

$$\phi(X) = (1 + X)^p - 1 = pX + \binom{p}{2}X^2 + \ldots + pX^{p-1} + X^p$$

is a Frobenius power series for p.

Lemma 1.20. *Let $\phi(X)$ and $\psi(X)$ be two Frobenius power series for π and let*

$$F_1(X_1, \ldots, X_n) = a_1X_1 + \ldots + a_nX_n \in \mathcal{O}_L[X_1, \ldots, X_n]$$

be a homogeneous linear form. There exists a unique formal power series $F(X_1, \ldots, X_n) \in \mathcal{O}_L[\![X_1, \ldots, X_n]\!]$ such that

$$F = F_1 + \textit{terms of degree } \geq 2 \qquad \textit{and} \qquad \phi(F(X_1, \ldots, X_n)) = F(\psi(X_1), \ldots, \psi(X_n)).$$

Proof. See Lemma 1.3.3 in [Sch17]. □

An important application of Lemma 1.20 is the following.

Proposition 1.21. *Let $\phi(X)$ be a Frobenius power series for π. Then there is a unique commutative formal group law $F_\phi(X, Y)$ over $\mathcal{O}_L$ such that $\phi \in \mathrm{End}(F_\phi)$.*

Proof. See Proposition 1.3.4 in [Sch17]. □

Definition 1.22. *F_ϕ is called the Lubin-Tate formal group law of the Frobenius power series ϕ.*

In other words, the Lubin-Tate formal group laws are the formal group laws admitting an endomorphism which modulo π becomes the q-th power Frobenius map and whose derivative at the origin equals the prime element π.

Example 1.23. (i) F_ϕ for $\phi(X) = \pi X + X^q$ is called the *special* Lubin-Tate group law of π.

(ii) If $L = \mathbb{Q}_p$ and $\pi = p$, one easily checks that $\phi(X) = (1 + X)^p - 1$ is an endomorphism of $\widehat{\mathbb{G}}_m$, therefore $F_\phi = \widehat{\mathbb{G}}_m$ by the uniqueness part of Proposition 1.21.

Another important application of Lemma 1.20 is the following.

Proposition 1.24. *Let $\phi(X)$ and $\psi(X)$ be two Frobenius power series for π. For each $a \in \mathcal{O}_L$ there exists a unique series $[a]_{\phi,\psi}(X) \in \mathcal{O}_L[\![X]\!]$ such that*

(i) $[a]_{\phi,\psi}(X) \in aX + X^2\mathcal{O}_L[\![X]\!]$,

(ii) $\phi([a]_{\phi,\psi}(X)) = [a]_{\phi,\psi}(\psi(X))$.

Moreover, $[a]_{\phi,\psi}(X)$ is a homomorphism $F_\psi \to F_\phi$.

Proof. See Proposition 1.3.6 in [Sch17]. □

In particular, when $\phi = \psi$ it follows that $[a]_\phi(X) := [a]_{\phi,\phi}(X)$ is an endomorphism of F_ϕ. By uniqueness we also obtain that $[1]_\phi(X) = X$ and $[\pi]_\phi(X) = \phi$. Furthermore, we can also show the following.

Lemma 1.25. *Let $\phi(X), \psi(X)$ and $\xi(X)$ be Frobenius power series for π. For any $a, b \in \mathcal{O}_L$ we have*

$$[a+b]_{\phi,\psi}(X) = [a]_{\phi,\psi}(X) +_{F_\phi} [b]_{\phi,\psi}(X),$$

$$[ab]_{\phi,\xi}(X) = [a]_{\phi,\psi}([b]_{\psi,\xi}(X)).$$

Proof. It is clear that

$$[a]_{\phi,\psi}(X) +_{F_\phi} [b]_{\phi,\psi}(X) = F_\phi([a]_{\phi,\psi}(X), [b]_{\phi,\psi}(X)) \in (a+b)X + X^2\mathcal{O}_L[\![X]\!],$$

hence by Proposition 1.24 it suffices to show that

$$\phi([a]_{\phi,\psi}(X) +_{F_\phi} [b]_{\phi,\psi}(X)) = [a]_{\phi,\psi}(\psi(X)) +_{F_\phi} [b]_{\phi,\psi}(\psi(X)).$$

A straighforward computation shows that

$$\begin{aligned}
\phi([a]_{\phi,\psi}(X) +_{F_\phi} [b]_{\phi,\psi}(X)) &= \phi(F_\phi([a]_{\phi,\psi}(X), [b]_{\phi,\psi}(X))) \\
&= F_\phi(\phi([a]_{\phi,\psi}(X)), \phi([b]_{\phi,\psi}(X))) \\
&= F_\phi([a]_{\phi,\psi}(\psi(X)), [b]_{\phi,\psi}(\psi(X))) \\
&= [a]_{\phi,\psi}(\psi(X)) +_{F_\phi} [b]_{\phi,\psi}(\psi(X)),
\end{aligned}$$

where in the second equality we used that $\phi \in \mathrm{End}(F_\phi)$ and in the third equality we used condition (ii) in Proposition 1.24.

Similarly, for multiplication, we first observe that

$$[a]_{\phi,\psi}([b]_{\psi,\xi}(X)) \in abX + X^2\mathcal{O}_L[\![X]\!],$$

therefore it suffices to show that

$$\phi([a]_{\phi,\psi}([b]_{\psi,\xi}(X))) = [a]_{\phi,\psi}([b]_{\psi,\xi}(\xi(X))).$$

For that we use condition (ii) in Lemma 1.24 twice to obtain that

$$\begin{aligned}
\phi([a]_{\phi,\psi}([b]_{\psi,\xi}(X))) &= [a]_{\phi,\psi}(\psi([b]_{\psi,\xi}(X))) \\
&= [a]_{\phi,\psi}([b]_{\psi,\xi}(\xi(X))),
\end{aligned}$$

as desired. $\square$

Remark 1.26. In particular, for any two Frobenius power series $\phi(X)$ and $\psi(X)$, their corresponding Lubin-Tate formal group laws F_ϕ and F_ψ are isomorphic. For every $u \in \mathcal{O}_L^\times$ the map

$$[u]_{\psi,\phi} : F_\phi \to F_\psi$$

is an isomorphism with inverse $[u^{-1}]_{\phi,\psi}$.

Corollary 1.27. *The map*

$$\begin{aligned} \mathcal{O}_L &\longrightarrow \mathrm{End}(F_\phi) \\ a &\longmapsto [a]_\phi(X) \end{aligned}$$

is an injective ring homomorphism.

Proof. The map is a ring homomorphism by Lemma 1.25 and it is injective because $[a]_\phi(X) \in aX + X^2\mathcal{O}_L[\![X]\!]$. □

Notation 1.28. For $a \in \mathcal{O}_L$ and arbitrary Frobenius power series ϕ and ψ, we let $\overline{[a]}_{\phi,\psi}(X)$ denotes the image of $[a]_{\phi,\psi}(X)$ in $\kappa_L[\![X]\!]$ with respect to the projection map.

Throughout the rest of the chapter, we fix a Frobenius power series $\phi(X) \in \mathcal{O}_L[\![X]\!]$ for our π and let F_ϕ denote its corresponding Lubin-Tate formal group law. We also fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ containing L. The field L is equipped with an absolute value $|\cdot|$, which extends uniquely to $\overline{\mathbb{Q}_p}$. We denote this extension by $|\cdot|$ as well. We let $\mathfrak{m}_{\overline{\mathbb{Q}_p}} := \{x \in \overline{\mathbb{Q}_p} : |x| < 1\}$.

As explained in Remark 1.15, for any subfield $E \subseteq \overline{\mathbb{Q}_p}$ that is a finite extension of L, we have the abelian group $(\mathfrak{m}_E, +_{F_\phi})$, where $\mathfrak{m}_E$ is the maximal ideal of the ring of integers of E. Since $\mathfrak{m}_{\overline{\mathbb{Q}_p}}$ is a union of such $\mathfrak{m}_E$, we also have the abelian group $(\mathfrak{m}_{\overline{\mathbb{Q}_p}}, +_{F_\phi})$. From Lemma 1.25 we furthermore know that $(\mathfrak{m}_{\overline{\mathbb{Q}_p}}, +_{F_\phi})$ is an $\mathcal{O}_L$-module via the multiplication

$$\begin{aligned} \mathcal{O}_L \times \mathfrak{m}_{\overline{\mathbb{Q}_p}} &\longrightarrow \mathfrak{m}_{\overline{\mathbb{Q}_p}} \\ (a, z) &\longmapsto [a]_\phi(z). \end{aligned}$$

For any $j \geq 1$ we have the $\mathcal{O}_L$-submodule

$$\mathcal{F}_j := \ker([\pi^j]_\phi) = \{z \in \mathfrak{m}_{\overline{\mathbb{Q}_p}} : [\pi^j]_\phi(z) = 0\}.$$

It is clear that $\mathcal{F}_j$ is an $\mathcal{O}_L/\pi^j\mathcal{O}_L$-module and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}_j \subseteq \ldots$. By adjoining these subsets of $\overline{\mathbb{Q}_p}$ to L we obtain the tower of algebraic extensions

$$L \subseteq L_1 := L(\mathcal{F}_1) \subseteq \ldots \subseteq L_j := L(\mathcal{F}_j) \subseteq \ldots \subseteq L_\infty := \bigcup_{j \geq 1} L_j \subseteq \overline{\mathbb{Q}_p}.$$

Example 1.29. Let $L = \mathbb{Q}_p$, $\pi = p$ and $\phi(X) = (1+X)^p - 1$, so that $F_\phi = \widehat{\mathbb{G}}_m$. Then

$$\mathcal{F}_j = \{\zeta - 1 : \zeta^{p^j} = 1\} \qquad \text{and} \qquad L_j = L(\{\zeta : \zeta^{p^j} = 1\}).$$

Remark 1.30. Remark 1.3.8 of [Sch17] shows that the extensions L_j and L_∞ only depend on π and not on the choice of the Frobenius power series ϕ.

Proposition 1.31. *(i) $\mathcal{F}_j$ is a free $\mathcal{O}_L/\pi^j\mathcal{O}_L$-module of rank one for every $j \geq 1$.*

(ii) There exists a sequence $(z_m)_{m\geq 1}$ such that z_m is a generator of $\mathcal{F}_m$ and $[\pi]_\phi(z_{m+1}) = z_m$ for every $m \geq 1$.

Proof. See Proposition 1.3.10 in [Sch17]. $\square$

Let $G_L = \mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ denote the absolute Galois group of L. Any automorphism $\sigma \in G_L$ respects the absolute value of $\overline{\mathbb{Q}_p}$. This implies that

$$\sigma([a]_\phi(z)) = [a]_\phi(\sigma(z))$$

for any $a \in \mathcal{O}_L$ and any $z \in \mathfrak{m}_{\overline{\mathbb{Q}_p}}$, and

$$\sigma(F_\phi(z_1, z_2)) = F_\phi(\sigma(z_1), \sigma(z_2))$$

for any $z_1, z_2 \in \mathfrak{m}_{\overline{\mathbb{Q}_p}}$. It follows that the Galois group G_L acts $\mathcal{O}_L/\pi^j\mathcal{O}_L$-linearly on $\mathcal{F}_j$ via

$$\begin{aligned} G_L \times \mathcal{F}_j &\longrightarrow \mathcal{F}_j \\ (\sigma, z) &\longmapsto \sigma(z). \end{aligned}$$

Together with Proposition 1.31, we deduce that L_j/L is a finite Galois extension and that for any $\sigma \in G_L$ there is a unique element $\chi_{L,j}(\sigma) \in (\mathcal{O}_L/\pi^j\mathcal{O}_L)^\times$ such that

$$\sigma(z) = [\chi_{L,j}(\sigma)]_\phi(z)$$

for any $z \in \mathcal{F}_j$, where the right hand side notation is justified since $[a]_\phi(z)$ only depends on a mod $\pi^j\mathcal{O}_L$ for $z \in \mathcal{F}_j$. An argument similar to that in Remark 1.30 shows that the elements $\chi_{L,j}(\sigma)$ only depend on π and not on the choice of ϕ.

Proposition 1.32. *For any $j \geq 1$, L_j/L is a totally ramified Galois extension of degree $(q-1)q^{j-1}$ and the map $\chi_{L,j} : G_L \to (\mathcal{O}_L/\pi^j\mathcal{O}_L)^\times$ restricts to an isomorphism of groups*

$$\chi_{L,j} : \mathrm{Gal}(L_j/L) \xrightarrow{\simeq} (\mathcal{O}_L/\pi^j\mathcal{O}_L)^\times.$$

Proof. See Proposition 1.3.12 in [Sch17]. $\square$

Let $H_L := \mathrm{Gal}(\overline{\mathbb{Q}_p}/L_\infty)$ and $\Gamma_L := G_L/H_L \simeq \mathrm{Gal}(L_\infty/L)$. The isomorphisms in Proposition 1.32 are compatible with respect to projections and passing to projective limits we obtain the isomorphism

$$\chi_L : \Gamma_L \xrightarrow{\simeq} \mathcal{O}_L^\times.$$

1.3 The field $\mathbb{C}_p$ and its tilt

We let $\mathbb{C}_p$ denote the completion of $\overline{\mathbb{Q}_p}$ with respect to $|\cdot|$. The absolute value $|\cdot|$ on $\overline{\mathbb{Q}_p}$ extends to $\mathbb{C}_p$ and we denote the extension by $|\cdot|$ as well. Whenever we refer to a topology on $\mathbb{C}_p$, we mean the one induced by the metric $|\cdot|$. For an intermediate field $\mathbb{Q}_p \subseteq K \subseteq \mathbb{C}_p$, we let $\mathcal{O}_K$ and $\mathfrak{m}_K$ denote the ring of integers of K and the maximal ideal of $\mathcal{O}_K$, respectively.

Before explaining the tilting construction, we first prove the following useful lemma.

Lemma 1.33. *Let B be a commutative $\mathcal{O}_L$-algebra and $a, b \in B$ such that $a - b \in \pi^m B$ for some $m \geq 1$. Then we have that*

$$a^{q^j} \equiv b^{q^j} \bmod \pi^{m+j} B$$

for every $j \geq 1$.

Proof. It suffices to prove that

$$a^q \equiv b^q \bmod \pi^{m+1} B.$$

Since $(a^p - b^p)|(a^q - b^q)$, it is enough to show that $a^p - b^p \in \pi^{m+1}B$. Write $a = b + \pi^m c$ for some $c \in B$. Then

$$a^p = (b + \pi^m c)^p = b^p + \sum_{k=1}^{p-1} \binom{p}{k} b^k (\pi^m c)^{p-k} + \pi^{mp} c^p \in b^p + \pi^{m+1} B$$

since $\pi | p$ and $mp \geq 2m \geq m + 1$ for $m \geq 1$. $\square$

The *tilt* of $\mathbb{C}_p$ is defined in the following way. We first let

$$\begin{aligned}\mathcal{O}_{\mathbb{C}_p^\flat} &:= \varprojlim \left(\dots \xrightarrow{(.)^q} \mathcal{O}_{\mathbb{C}_p}/\pi\mathcal{O}_{\mathbb{C}_p} \xrightarrow{(.)^q} \mathcal{O}_{\mathbb{C}_p}/\pi\mathcal{O}_{\mathbb{C}_p} \xrightarrow{(.)^q} \dots \xrightarrow{(.)^q} \mathcal{O}_{\mathbb{C}_p}/\pi\mathcal{O}_{\mathbb{C}_p} \right) \\ &= \left\{ (\dots, y_j, \dots, y_1, y_0) \in \left(\mathcal{O}_{\mathbb{C}_p}/\pi\mathcal{O}_{\mathbb{C}_p}\right)^{\mathbb{N}_{\geq 0}} : y_{j+1}^q = y_j \text{ for any } j \geq 0 \right\}.\end{aligned}$$

Then, as the q-th power map is a κ_L-algebra endomorphism of $\mathcal{O}_{\mathbb{C}_p}/\pi\mathcal{O}_{\mathbb{C}_p}$ we see that $\mathcal{O}_{\mathbb{C}_p^\flat}$ is a κ_L-algebra.

Let $y = (\dots, y_j, \dots, y_0)$ be an element of $\mathcal{O}_{\mathbb{C}_p^\flat}$. For any $j \geq 0$ we choose an element $a_j \in \mathcal{O}_{\mathbb{C}_p}$ such that $a_j \bmod \pi\mathcal{O}_{\mathbb{C}_p} = y_j$. Then $a_{j+1}^q \equiv a_j \bmod \pi\mathcal{O}_{\mathbb{C}_p}$ and hence by Lemma 1.33 we have that $a_{j+1}^{q^{j+1}} \equiv a_j^{q^j} \bmod \pi^{j+1}\mathcal{O}_{\mathbb{C}_p}$. It follows that the limit

$$y^\sharp := \lim_{j\to\infty} a_j^{q^j} \in \mathcal{O}_{\mathbb{C}_p}$$

exists and another application of Lemma 1.33 shows that $y^\sharp$ is independent of the choice of the a_j. We define

$$\begin{aligned} |\cdot|_\flat : \mathcal{O}_{\mathbb{C}_p^\flat} &\longrightarrow \mathbb{R}_{\geq 0} \\ y &\longmapsto \left|y^\sharp\right|. \end{aligned}$$

In the statements of Remark 1.4.4 − Lemma 1.4.6 in [Sch17] it is proven that $\mathcal{O}_{\mathbb{C}_p^\flat}$ is a perfect integral domain and $|\cdot|_\flat$ is a multiplicative nonarchimedean absolute value on $\mathcal{O}_{\mathbb{C}_p^\flat}$. It follows that $|\cdot|_\flat$ extends to

$$\mathbb{C}_p^\flat := \mathrm{Frac}(\mathcal{O}_{\mathbb{C}_p^\flat}).$$

Lemma 1.34. *$\mathbb{C}_p^\flat$ is a perfect and complete nonarchimedean field of characteristic p with respect to $|\cdot|_\flat$.*

Proof. See Proposition 1.4.7 of [Sch17]. □

Lemma 1.35. *The field $\mathbb{C}_p^\flat$ is algebraically closed.*

Proof. See Lemma 1.4.10 of [Sch17]. □

The group G_L acts continuously on $\mathbb{C}_p$ and every element of G_L preserves the absolute value in $\mathbb{C}_p$. In particular, the G_L-action on $\mathbb{C}_p$ preserves $\pi\mathcal{O}_{\mathbb{C}_p}$ and hence induces an action

$$\begin{aligned} G_L \times \mathcal{O}_{\mathbb{C}_p^\flat} &\to \mathcal{O}_{\mathbb{C}_p^\flat} \\ \big(\sigma, \big(\ldots, a_j \bmod \pi\mathcal{O}_{\mathbb{C}_p}, \ldots\big)\big) &\longmapsto \big(\ldots, \sigma(a_j) \bmod \pi\mathcal{O}_{\mathbb{C}_p}, \ldots\big) \end{aligned} \tag{1.2}$$

which extends uniquely to an action

$$G_L \times \mathbb{C}_p^\flat \longrightarrow \mathbb{C}_p^\flat.$$

By the continuity of the action of G_L on $\mathbb{C}_p$, one has that

$$\sigma(y^\sharp) = (\sigma(y))^\sharp$$

for any $\sigma \in G_L$ and $y \in \mathcal{O}_{\mathbb{C}_p^\flat}$. It follows that the action (1.2) preserves the absolute value $|\cdot|_\flat$ on $\mathcal{O}_{\mathbb{C}_p^\flat}$, from which we can also conclude the following.

Lemma 1.36. *Every element of G_L preserves the absolute value of $\mathbb{C}_p^\flat$.*

Furthermore, we also have that:

Proposition 1.37. *G_L acts continuously on $\mathbb{C}_p^\flat$ for the topology defined by $|\cdot|_\flat$.*

Proof. See Lemma 1.4.13 in [Sch17]. □

Consider the field of Laurent series in one variable $E_L := \kappa_L((X))$ and let $(z_m)_{m\geq 1}$ be a sequence such that $z_m \in \mathcal{F}_m$ is a generator of $\mathcal{F}_m$ and $[\pi]_\phi(z_{m+1}) = z_m$ for all $m \geq 1$. The existence of such a sequence is guaranteed by Proposition 1.31 (ii). We embed E_L into $\mathbb{C}_p^\flat$ by sending X to

$$\omega := (\ldots, z_m \bmod \pi\mathcal{O}_{\mathbb{C}_p}, \ldots, z_1 \bmod \pi\mathcal{O}_{\mathbb{C}_p}, 0) \in \mathcal{O}_{\mathbb{C}_p^\flat}.$$

Lemma 1.38. *(i) We have that $|\omega|_\flat = |\pi|^{\frac{q}{q-1}} < 1$.*

(ii) The image of E_L in $\mathbb{C}_p^\flat$ does not depend on the choice of the sequence $(z_m)_{m\geq 1}$ and for $\sigma \in G_L$, we have the equality

$$\sigma(\omega) = \overline{[\chi_L(\overline{\sigma})]_\phi}(\omega)$$

where $\overline{\sigma} := \sigma \bmod H_L$.

(iii) The group G_L preserves the image of E_L.

Proof. Part (i) is proven in Lemma 1.4.14 of [Sch17], while Lemma 1.4.15 of [Sch17] proves part (ii). The last part then follows from (ii) and Proposition 1.37. □

Since $\mathbb{C}_p^\flat$ is algebraically closed, the above embedding extends to an embedding from $\overline{E_L}$ into $\mathbb{C}_p^\flat$. Let E_L^{sep} denote a separable closure of E_L. From now on we identify $\overline{E_L}$ and E_L^{sep} with their images in $\mathbb{C}_p^\flat$. We also let

$$E_L^{\mathrm{sep}\,+} := \{x \in E_L^{\mathrm{sep}} : |x|_\flat \leq 1\}$$

denote the ring of integers of E_L^{sep}.

Lemma 1.39. *The action of the group G_L on $\mathbb{C}_p^\flat$ preserves E_L^{sep} and $E_L^{\mathrm{sep}\,+}$.*

Proof. Let $y \in E_L^{\mathrm{sep}}$ and $\sigma \in G_L$. Suppose that

$$P(T) = T^m + e_{m-1}T^{m-1} + \ldots + e_1 T + e_0$$

is the minimal polynomial of y over E_L. Then

$$Q(T) = T^m + \sigma(e_{m-1})T^{m-1} + \ldots + \sigma(e_1)T + \sigma(e_0)$$

is a polynomial in $E_L[T]$ which vanishes on $\sigma(y)$. In fact $Q(T)$ is the minimal polynomial of $\sigma(y)$ over E_L, because σ and σ^{-1} induce degree preserving automorphisms on $E_L[T]$. To show that $Q(T)$ is separable, note that the separability of $P(T)$ implies that $P(T)$ is not a polynomial in T^p. Therefore neither is $Q(T)$ because σ restricts to an automorphism of E_L. Thus $\sigma(y) \in E_L^{\mathrm{sep}}$, as desired. Since every automorphism $\sigma \in G_L$ preserves the absolute value $|\cdot|_\flat$ on $\mathbb{C}_p^\flat$, it follows that the action of G_L preserves $E_L^{\mathrm{sep}\,+}$ as well. □

We conclude with a modernized version of the theorem of Fontaine and Wintenberger [FW79], which is proven in Section 1.6 of [Sch17] using the tilting correspondence.

Proposition 1.40. *The action of H_L on E_L^{sep} induces an isomorphism of groups*

$$H_L \simeq \mathrm{Gal}(E_L^{\mathrm{sep}}/E_L).$$

1.4 The linear topology of modules over a topological ring

Let R be a commutative ring with unity equipped with a topology for which it is a topological ring. In this section we explain how we can naturally associate a topology to a finitely generated R-module. For $k \in \mathbb{N}_{\geq 1}$, equip $R^{\oplus k}$ with the product topology.

Lemma 1.41. *For $\ell, m \in \mathbb{N}_{\geq 1}$, every R-linear map*

$$\psi : R^{\oplus \ell} \to R^{\oplus m}$$

is continuous.

Proof. Let $(a_1, \ldots, a_\ell) \in R^{\oplus \ell}$ and $C = (c_{jk})_{\substack{1 \leq j \leq m \\ 1 \leq k \leq \ell}}$ be the matrix of ψ with respect to the standard bases on both sides. Let $V \subseteq R$ be a basic open neighbourhood of zero. Since addition on R is continuous, we can choose an open neighbourhood U' of zero such that

$$\sum_{k=1}^{\ell} (c_{jk} a_k + U') \subseteq \sum_{k=1}^{\ell} c_{jk} a_k + V$$

for all $1 \leq j \leq m$. Since multiplication on R is continuous, we can also choose an open neighbourhood U of zero such that

$$c_{jk} (a_k + U) \subseteq c_{jk} a_k + U'$$

for all $1 \leq j \leq m$ and $1 \leq k \leq \ell$. Then clearly

$$\begin{aligned} \psi (a_1 + U, \ldots, a_\ell + U) &= \left(\sum_{k=1}^{\ell} c_{1k}(a_k + U), \ldots, \sum_{k=1}^{\ell} c_{mk}(a_k + U) \right) \\ &= \left(\sum_{k=1}^{\ell} (c_{1k} a_k + U'), \ldots, \sum_{k=1}^{\ell} (c_{mk} a_k + U') \right) \\ &\subseteq \left(\sum_{k=1}^{\ell} c_{1k} a_k + V, \ldots, \sum_{k=1}^{\ell} c_{mk} a_k + V \right), \end{aligned}$$

where the last equality holds because V is closed under addition. □

Let D be an arbitrary finitely generated R-module. Consider any R-linear surjection $\psi : R^{\oplus k} \twoheadrightarrow D$ and equip D with the quotient topology, meaning that $U \subseteq D$ is defined to be open precisely when $\psi^{-1}(U) \subseteq R^{\oplus k}$ is open. The definition of the topology does not depend on the chosen $k \in \mathbb{N}_{>0}$ and surjection ψ. Indeed, suppose that we have two arbitrary surjections

$$\begin{tikzcd} R^{\oplus \ell} \arrow[dr, two heads, "\psi_1"] & & R^{\oplus m} \arrow[dl, two heads, "\psi_2"'] \\ & D & \end{tikzcd}$$

then as $R^{\oplus \ell}$ and $R^{\oplus m}$ are projective modules over R, there exist R-linear maps $f_1 : R^{\oplus \ell} \to R^{\oplus m}$ and $f_2 : R^{\oplus m} \to R^{\oplus \ell}$ such that

$$\begin{tikzcd} R^{\oplus \ell} \arrow[rr, "f_1"] \arrow[dr, "\psi_1"'] & & R^{\oplus m} \arrow[dl, "\psi_2"] \\ & D & \end{tikzcd} \qquad \begin{tikzcd} R^{\oplus \ell} \arrow[dr, "\psi_1"'] & & R^{\oplus m} \arrow[ll, "f_2"'] \arrow[dl, "\psi_2"] \\ & D & \end{tikzcd}$$

commute. By Lemma 1.41, f_1 and f_2 are continuous and the conclusion follows.

We call the topology of D obtained in this way the *linear topology.*

Corollary 1.42. *Let D_1 and D_2 be finitely generated R-modules equipped with the linear topology. Then every R-linear map*

$$f : D_1 \to D_2$$

is continuous.

Proof. Consider two R-linear surjections $\psi_1 : R^{\oplus \ell} \twoheadrightarrow D_1$ and $\psi_2 : R^{\oplus m} \twoheadrightarrow D_2$. Because $R^{\oplus \ell}$ is projective, we know that there exists an R-linear map

$$\psi : R^{\oplus \ell} \longrightarrow R^{\oplus m}$$

such that

$$\begin{tikzcd} R^{\oplus \ell} \arrow[r, "\psi"] \arrow[d, two heads, "\psi_1"] & R^{\oplus m} \arrow[d, two heads, "\psi_2"] \\ D_1 \arrow[r, "f"] & D_2 \end{tikzcd} \tag{1.3}$$

commutes. Let $U \subseteq D_2$ be an open set. To show that $f^{-1}(U) \subseteq D_1$ is open is equivalent to showing that $\psi_1^{-1}(f^{-1}(U)) \subseteq R^{\oplus \ell}$ is open, which by the commutativity of (1.3) equals $\psi^{-1}(\psi_2^{-1}(U))$. Since ψ is continuous by Lemma 1.41 and ψ_2 is continuous by definition, the conclusion follows. $\square$

For a finitely generated R-module D, any R-linear surjection $\psi : R^{\oplus k} \twoheadrightarrow D$ in addition to being a quotient map by definition, is actually an open map as well.

Lemma 1.43. *Let D be a finitely generated R-module and $\psi : R^{\oplus k} \twoheadrightarrow D$ be an R-linear surjection. Then ψ is open with respect to the linear topology.*

Proof. Let $U \subseteq R^{\oplus k}$ be an open set. To show that $\psi(U) \subseteq D$ is open means to show that $\psi^{-1}(\psi(U)) \subseteq R^{\oplus k}$ is open. We know that

$$\psi^{-1}(\psi(U)) = \bigcup_{x \in \ker(\psi)} x + U$$

and each $x + U$ is open because addition is continuous on $R^{\oplus k}$ for the linear topology due to continuity of addition on R. $\square$

Lemma 1.44. *Let D_1 and D_2 be finitely generated R-modules. Then the linear topology on $D_1 \oplus D_2$ coincides with the product of the linear topologies of D_1 and D_2.*

Proof. Consider two R-linear surjections $\psi_1 : R^{\oplus \ell} \twoheadrightarrow D_1$ and $\psi_2 : R^{\oplus m} \twoheadrightarrow D_2$. Then

$$\psi_1 \oplus \psi_2 : R^{\oplus \ell + m} \twoheadrightarrow D_1 \oplus D_2$$

is also an R-linear surjection. If $U \subseteq D_1$ and $V \subseteq D_2$ are open, then

$$(\psi_1 \oplus \psi_2)^{-1}(U \times V) = \psi_1^{-1}(U) \times \psi_2^{-1}(V) \subseteq R^{\oplus \ell + m}$$

is open, therefore $U \times V$ is open for the linear topology on $D_1 \oplus D_2$. Conversely, let $W \subseteq D_1 \oplus D_2$ be open for the linear topology on $D_1 \oplus D_2$. Then $(\psi_1 \oplus \psi_2)^{-1}(W) \subseteq R^{\oplus \ell + m}$ is open. The linear topology of $R^{\oplus \ell + m}$ equals the product of the linear topologies of $R^{\oplus \ell}$ and $R^{\oplus m}$ by definition. Therefore

$$(\psi_1 \oplus \psi_2)^{-1}(W) = \bigcup_{j \in J} U_j \times V_j$$

where $U_j \subseteq R^{\oplus \ell}$ and $V_j \subseteq R^{\oplus m}$ are open. By Lemma 1.43 $\psi_1(U_j) \subseteq D_1$ and $\psi_2(V_j) \subseteq D_2$ are open for every $j \in J$. Then

$$W = (\psi_1 \oplus \psi_2) \circ ((\psi_1 \oplus \psi_2)^{-1}(W)) = \bigcup_{j \in J} \psi_1(U_j) \times \psi_2(V_j)$$

is open for the product of the linear topologies on D_1 and D_2, as desired. $\square$

Lemma 1.45. *D is a topological R-module for the linear topology.*

Proof. The addition map $D \oplus D \to D$ is R-linear, therefore combining Corollary 1.42 and Lemma 1.44 gives us that addition on D is continuous.

For scalar multiplication, consider a surjection $\psi : R^{\oplus k} \twoheadrightarrow D$ and the commutative diagram

$$\begin{tikzcd}
R \times R^{\oplus k} \arrow[r, "\rho"] \arrow[d, "\mathrm{id} \times \psi"] & R^{\oplus k} \arrow[d, "\psi"] \\
R \times D \arrow[r, "\rho_D"] & D
\end{tikzcd}$$

whose horizontal arrows are the scalar multiplication maps. Let $U \subseteq D$ be an open set. The left vertical arrow is a continuous surjective open map by Lemma 1.43, thus a quotient map. Then to show that $\rho_D^{-1}(U) \subseteq R \times D$ is open, we need to show that $(\mathrm{id} \times \psi)^{-1}(\rho_D^{-1}(U)) \subseteq R \times R^{\oplus k}$ is open. We know that

$$(\mathrm{id} \times \psi)^{-1}(\rho_D^{-1}(U)) = \rho^{-1}(\psi^{-1}(U))$$

and the conclusion follows because ψ is continuous and ρ is continuous due to continuity of multiplication on R. $\square$

For convenience, we generalize the result of Lemma 1.43 using a similar proof.

Lemma 1.46. *Suppose that D_1 and D_2 are finitely generated R-modules equipped with the linear topology. If $f : D_1 \twoheadrightarrow D_2$ is an R-linear surjection, then f is open.*

Proof. Let $U \subseteq D_1$ open. Let $\psi : R^{\oplus k} \twoheadrightarrow D_1$ be an R-linear surjection. Then

$$f \circ \psi : R^{\oplus k} \twoheadrightarrow D_2$$

is an R-linear surjection as well. To show that $f(U) \subseteq D_2$ is open, we need to show that $(f \circ \psi)^{-1}(f(U)) = \psi^{-1}(f^{-1}(f(U))) \subseteq R^{\oplus k}$ is open.

$$f^{-1}(f(U)) = \bigcup_{x \in \ker f} (x + U).$$

Each $x + U$ is open by Lemma 1.45, thus $f^{-1}(f(U)) \subseteq D_1$ is open. Since ψ is continuous, the conclusion follows. □

1.5 Completeness and a version of Nakayama Lemma

In this section, we state some results from commutative algebra that we will use in the later chapters. We begin with a straightforward lemma about a situation when taking completions commutes with taking tensor products.

Lemma 1.47. *Let R be an arbitrary commutative ring, I an ideal of R and N an I-adically complete R-module.*

(i) If R is I-adically complete, then

$$N \otimes_R \varprojlim_m M/I^m M \simeq \varprojlim_m (N \otimes_R M/I^m M)$$

for any finite free R-module M.

(ii) If R is a complete DVR with uniformizer δ and $I = \delta R$, then

$$N \otimes_R \varprojlim_m M/I^m M \simeq \varprojlim_m (N \otimes_R M/I^m M)$$

for any finitely generated R-module M.

Proof. (i) Each of the involved functors is additive, therefore our claim reduces to the case when $M = R$. The ring R is I-adically complete, hence

$$N \otimes_R \varprojlim_m R/I^m R \simeq N \otimes_R R \simeq N.$$

In addition, the module N is I-adically complete, which implies that

$$\varprojlim_m (N \otimes_R R/I^m R) \simeq \varprojlim_m N/I^m N \simeq N.$$

(ii) Using the elementary divisor theorem and that limits are additive, it suffices to check the case when $M = R$ or $M = R/\delta^s R$ for $s \in \mathbb{N}_{\geq 1}$. When $M = R$, the conclusion follows from part (i). When $M = R/\delta^s R$ for some $s \in \mathbb{N}_{\geq 1}$, we have that $\delta^{s+c} M = 0$ for every $c \in \mathbb{N}_{\geq 0}$, therefore $\varprojlim_m M/\delta^m M \simeq M$ and $\varprojlim_m (N \otimes_R M/\delta^m M) \simeq \varprojlim_m N \otimes_R M \simeq N \otimes_R M$ and the conclusion follows. $\square$

In other words, Lemma 1.47 shows that $M \otimes_R N$ is I-adically complete under the stated assumptions. We can also merely assume that R is Noetherian and that M is finitely generated over R, if we additionally require that N is flat over R.

Lemma 1.48. *Let R be a Noetherian ring and I an ideal of R. Suppose that M is a finitely generated R-module. Suppose that N is a flat R-module which is I-adically complete. Then $M \otimes_R N$ is I-adically complete.*

Proof. We start with a short exact sequence

$$0 \longrightarrow K \longrightarrow R^{\oplus \ell} \longrightarrow M \longrightarrow 0$$

where $\ell \in \mathbb{N}_{\geq 1}$. By the flatness of N over R, the sequence

$$0 \longrightarrow K_N \longrightarrow N^{\oplus \ell} \longrightarrow M_N \longrightarrow 0$$

is exact where $(-)_N := - \otimes_R N$. This induces the exact sequences

$$0 \longrightarrow K_N/(I^m N^{\oplus \ell} \cap K_N) \longrightarrow N^{\oplus \ell}/I^m N^{\oplus \ell} \longrightarrow M_N/I^m M_N \longrightarrow 0$$

for every $m \in \mathbb{N}_{\geq 1}$. As the terms on the left satisfy the Mittag-Leffler property, the sequence

$$0 \longrightarrow \varprojlim_m K_N/(I^m N^{\oplus \ell} \cap K_N) \longrightarrow \varprojlim_m N^{\oplus \ell}/I^m N^{\oplus \ell} \longrightarrow \varprojlim_m M_N/I^m M_N \longrightarrow 0$$

is exact. The ring R is Noetherian, therefore by the Artin-Rees Lemma, there exists a $c \in \mathbb{N}_{\geq 0}$ such that

$$I^m K \subseteq I^m R^{\oplus \ell} \cap K \subseteq I^{m-c} K \tag{1.4}$$

for all $m \geq c$. Using the flatness of N over R, applying $(-)_N$ to (1.4), we obtain that the natural maps

$$(I^m K)_N \hookrightarrow (I^m R^{\oplus \ell} \cap K)_N \hookrightarrow (I^{m-c} K)_N \tag{1.5}$$

are injective. Because N is flat over R, we have that $(I^m K)_N$ is the kernel of the map

$$K_N \longrightarrow (K/I^m K)_N. \tag{1.6}$$

Also, since $(K/I^m K)_N \simeq K_N/I^m K_N$, we also know that $I^m K_N$ is the kernel of (1.6), hence both $I^m K_N$ and $(I^m K)_N$ are identified as submodules of $N^{\oplus \ell}$. Similarly, $I^{m-c}K_N$ and $(I^{m-c}K)_N$ are identified as submodules of $N^{\oplus \ell}$.

Furthermore, since N is flat over R, we have that $(I^m R^{\oplus \ell} \cap K)_N$ is the kernel of the map

$$N^{\oplus \ell} \longrightarrow (R^{\oplus \ell}/I^m R^{\oplus \ell} \oplus R^{\oplus \ell}/K)_N. \tag{1.7}$$

Using the isomorphism

$$(R^{\oplus \ell}/I^m R^{\oplus \ell} \oplus R^{\oplus \ell}/K)_N \simeq N^{\oplus \ell}/I^m N^{\oplus \ell} \oplus N^{\oplus \ell}/K_N,$$

we also know that $I^m N^{\oplus \ell} \cap K_N$ is the kernel of (1.7), therefore $(I^m R^{\oplus \ell} \cap K)_N$ and $I^m N^{\oplus \ell} \cap K_N$ are identified as submodules of $N^{\oplus \ell}$. Using these identifications, the embeddings in (1.5) become the inclusions

$$I^m K_N \subseteq I^m N^{\oplus \ell} \cap K_N \subseteq I^{m-c} K_N.$$

Therefore we have that

$$\varprojlim_m K_N/(I^m N^{\oplus \ell} \cap K_N) \simeq \varprojlim_m K_N/I^m K_N.$$

We then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_N \arrow[r] \arrow[d] & N^{\oplus \ell} \arrow[r] \arrow[d] & M_N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim_m K_N/I^m K_N \arrow[r] & \varprojlim_m N^{\oplus \ell}/I^m N^{\oplus \ell} \arrow[r] & \varprojlim_m M_N/I^m M_N \arrow[r] & 0
\end{tikzcd} \tag{1.8}$$

whose rows are exact. The middle arrow is bijective as N is I-adically complete, therefore the natural map

$$M_N \longrightarrow \varprojlim_m M_N/I^m M_N$$

is surjective. The Noetherianity of R allows us to redo the above argument for K to show that the natural map

$$K_N \longrightarrow \varprojlim_m K_N/I^m K_N$$

is surjective as well. Therefore using the snake lemma in (1.8) we conclude that its right vertical arrow is bijective, in other words that $M \otimes_R N$ is I-adically complete. $\square$

Another important fact is that the ideal adic completion of a flat module over a Noetherian ring remains flat. This will be a consequence of [Sta18, Tag 0AGW] whose statement we write here:

Lemma 1.49. *Let R be a commutative ring and I an ideal of R. Let M be an R-module. Assume that*

1. *I is finitely generated,*
2. *R/I is Noetherian,*
3. *M/IM is flat over R/I,*
4. *$\operatorname{Tor}_1^R(M, R/I) = 0$.*

Then the I-adic completion $\widehat{R}$ is a Noetherian ring and the I-adic completion $\widehat{M}$ is flat over $\widehat{R}$.

Corollary 1.50. *Let R be a Noetherian ring, $I \trianglelefteq R$ an ideal and M a flat R-module. Then the I-adic completion*

$$\widehat{M} := \varprojlim_m M/I^m M$$

is a flat R-module.

Proof. This is Theorem 0.1 in [Yek18]. Alternatively, the Noetherianity of R and the flatness of M over R imply that the conditions of Lemma 1.49 are satisfied, which shows that $\widehat{M}$ is flat over $\widehat{R}$. We also know that $\widehat{R}$ is flat over R by [Sta18, Tag 00MB], thus

$$\widehat{M} \otimes_R - \simeq \widehat{M} \otimes_{\widehat{R}} (\widehat{R} \otimes_R -)$$

preserves injectivity. □

We end this section with a useful version of Nakayama Lemma in the setting of modules over adically complete rings whose proof is provided in Theorem 8.4 of [Mat89]. Its main advantage is that it does not apriori require the module in question to be finitely generated.

Lemma 1.51. *Let R be a commutative ring, I an ideal of R and M an R-module. Suppose that R is I-adically complete and that M is I-adically Hausdorff in the sense that*

$$\bigcap_{k \geq 1} I^k M = 0.$$

If M/IM is finitely generated over R/I, then M is finitely generated over R. More precisely, if $x_1, \ldots, x_\ell$ are in M such that their images $\overline{x}_1, \ldots, \overline{x}_\ell$ in M/IM generate M/IM over R/I, then $x_1, \ldots, x_\ell$ generate M over R.

Chapter 2

The categories

In this chapter we introduce the relevant categories for our thesis. One of them is the category of continuous representations of the n-fold product of G_L with coefficients in $\mathcal{O}_L$. For the other category, we first define the coefficient ring $\mathcal{A}_{\Delta_n}$ in Section 2.1, while in Section 2.2 we explore some important properties of the quotient ring $\mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n}$. The ring $\mathcal{A}_{\Delta_n}$ comes equipped with an action of the n-fold product of Γ_L and with n partial Frobenius maps. The category of multivariable (φ, Γ)-modules is the category of finitely generated modules over $\mathcal{A}_{\Delta_n}$ which are equipped with a semilinear action of the n-fold product of Γ_L and n semilinear partial Frobenius maps. We will be particularly interested in the multivariable (φ, Γ)-modules that are *étale*. Such modules additionally require that the linearized versions of the partial Frobenii are isomorphisms.

In the definition of our category of multivariable (φ, Γ)-modules we do not impose any continuity condition, as opposed to our category of Galois representations. In Section 2.3 we introduce two relevant topologies on $\mathcal{A}_{\Delta_n}$, hence by the theory of Section 1.4 we obtain two topologies on a finitely generated $\mathcal{A}_{\Delta_n}$-module, which we will call the adic and the weak topology. In Section 2.5 we prove that if our multivariable (φ, Γ)-module is étale, then the underlying action of the n-fold product of Γ_L is always continuous for the weak topology on our underlying multivariable (φ, Γ)-module.

2.1 The rings $\mathcal{A}_{\Delta_n}$ and E_{Δ_n}

For $n \in \mathbb{N}_{>0}$ we denote the set $\{1, \dots, n\}$ by Δ_n and we let $G_{i,L}, H_{i,L}, \Gamma_{i,L}$ denote copies of G_L, H_L and Γ_L, respectively, indexed by $i \in \Delta_n$. Let

$$G_{\Delta_n,L} \coloneqq \prod_{i\in\Delta_n} G_{i,L},$$

$$H_{\Delta_n,L} \coloneqq \prod_{i\in\Delta_n} H_{i,L}$$

and

$$\Gamma_{\Delta_n,L} \coloneqq \prod_{i\in\Delta_n} \Gamma_{i,L}.$$

For each $i \in \Delta_n$, we also fix a Frobenius power series $\phi_i(X)$ for π. Consider the multivariable power series ring $\mathcal{O}^+_{\Delta_n} \coloneqq \mathcal{O}_L[\![X_1,\dots,X_n]\!]$ and its localization $\mathcal{O}_{\Delta_n} \coloneqq \mathcal{O}^+_{\Delta_n}[X^{-1}_{\Delta_n}]$, where for each $S \subseteq \Delta_n$

$$X_S \coloneqq \prod_{i\in S} X_i.$$

Further, we also consider the π-adic completion

$$\begin{aligned}\mathcal{A}_{\Delta_n} :&= \varprojlim_m \mathcal{O}_{\Delta_n}/\pi^m\mathcal{O}_{\Delta_n}\\ &= \varprojlim_m \mathcal{O}_L/\pi^m\mathcal{O}_L[\![X_1,\dots,X_n]\!][X^{-1}_{\Delta_n}].\end{aligned}$$

This is a multivariable analog of the $\mathcal{O}_L$-algebra

$$\mathcal{A}_L \coloneqq \varprojlim_m \mathcal{O}_L(\!(X)\!)/\pi^m\mathcal{O}_L(\!(X)\!) = \varprojlim_m \mathcal{O}_L/\pi^m\mathcal{O}_L(\!(X)\!).$$

We can also identify the elements of $\mathcal{A}_{\Delta_n}$ with Laurent series

$$\sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} c_{i_1,\dots,i_n} X_1^{i_1}\dots X_n^{i_n}$$

with coefficients in $\mathcal{O}_L$ for which $\lim\limits_{\min(i_1,\dots,i_n)\to-\infty} c_{i_1,\dots,i_n} = 0.$

Indeed, if $\sum\limits_{(i_1,\dots,i_n)\in\mathbb{Z}^n} c_{i_1,\dots,i_n} X_1^{i_1}\dots X_n^{i_n}$ is such a Laurent series, then

$$\left(\sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} (c_{i_1,\dots,i_n} \bmod \pi^m\mathcal{O}_L) X_1^{i_1}\dots X_n^{i_n}\right)_m \in \varprojlim_m \mathcal{O}_{\Delta_n}/\pi^m\mathcal{O}_{\Delta_n} = \mathcal{A}_{\Delta_n}.$$

Conversely, let

$$\left(\sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} (c_{i_1,\dots,i_n,m} \bmod \pi^m\mathcal{O}_L) X_1^{i_1}\dots X_n^{i_n}\right)_m \in \mathcal{A}_{\Delta_n}$$

for some $c_{i_1,\dots,i_n,m} \in \mathcal{O}_L$. Then

$$c_{i_1,\dots,i_n,m+1} - c_{i_1,\dots,i_n,m} \in \pi^m\mathcal{O}_L$$

for all $m \geq 1$, therefore $(c_{i_1,\dots,i_n,m})_{m\geq 1}$ is a Cauchy sequence. Let

$$c_{i_1,\dots,i_n} \coloneqq \lim_{m\to\infty} c_{i_1,\dots,i_n,m}$$

and take $\sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} c_{i_1,\dots,i_n} X_1^{i_1}\dots X_n^{i_n}$. Then, given any m, there exists a sufficiently small value of $\min(j_1,\dots,j_n)$ such that $c_{j_1,\dots,j_n,m} \bmod \pi^m\mathcal{O}_L = 0$. Then $c_{j_1,\dots,j_n} \in \pi^m\mathcal{O}_L$ and our series satisfies the desired property.

Lemma 2.1. *$\mathcal{A}_{\Delta_n}$ is a π-adically complete Noetherian ring.*

Proof. The ring $\mathcal{O}^+_{\Delta_n}$ is Noetherian by the Noetherianity of $\mathcal{O}_L$ and the formal Hilbert basis theorem. Therefore $\mathcal{O}_{\Delta_n}$ is Noetherian as well. By [Sta18, Tag 0316] $\mathcal{A}_{\Delta_n}$ is then Noetherian and π-adically complete as well. □

For $i \in \Delta_n$, we let

$$\mathcal{A}_i \coloneqq \varprojlim_m \mathcal{O}_L(\!(X_i)\!)/\pi^m\mathcal{O}_L(\!(X_i)\!)$$

be a copy of $\mathcal{A}_L$ whose free variable is X_i. By Lemma 1.7.1 of [Sch17], $\mathcal{A}_i$ is a complete DVR with uniformizer π, therefore $[b]_{\phi_i}(X_i) \in \mathcal{A}_i^\times \cap X_i\mathcal{O}_L[\![X_i]\!]$ for every $b \in \mathcal{O}_L^\times$. We also know that $[\pi]_{\phi_i}(X_i) \in \mathcal{A}_i^\times \cap X_i\mathcal{O}_L[\![X_i]\!]$ because $[\pi]_{\phi_i}(X_i) \equiv X_i^q \bmod \pi\mathcal{O}_L[\![X_i]\!]$. Therefore, the maps

$$\begin{aligned} \mathcal{A}_i &\longrightarrow \mathcal{A}_i \\ \sum_{j\in\mathbb{Z}} a_j X_i^j &\longmapsto \sum_{j\in\mathbb{Z}} a_j \left([b]_{\phi_i}(X_i)\right)^j \end{aligned}$$

and

$$\begin{aligned} \varphi_i : \mathcal{A}_i &\longrightarrow \mathcal{A}_i \\ \sum_{j\in\mathbb{Z}} a_j X_i^j &\longmapsto \sum_{j\in\mathbb{Z}} a_j \left([\pi]_{\phi_i}(X_i)\right)^j \end{aligned}$$

are well-defined $\mathcal{O}_L$-algebra endomorphisms, where $a_j, b \in \mathcal{O}_L^\times$. Using the inclusions $\mathcal{A}_i \subseteq \mathcal{A}_{\Delta_n}$, these maps enable us to define an action of $\Gamma_{\Delta_n,L}$ on $\mathcal{A}_{\Delta_n}$

$$\begin{aligned} \Gamma_{\Delta_n,L} \times \mathcal{A}_{\Delta_n} &\longrightarrow \mathcal{A}_{\Delta_n} \\ ((\sigma_1,\dots,\sigma_n), f) &\longmapsto (\sigma_1,\dots,\sigma_n)\cdot f \coloneqq f\left([\chi_L(\sigma_1)]_{\phi_1}(X_1),\dots,[\chi_L(\sigma_n)]_{\phi_n}(X_n)\right) \end{aligned}$$

together with $\mathcal{O}_L$-algebra endomorphisms

$$\begin{aligned} \varphi_i : \mathcal{A}_{\Delta_n} &\longrightarrow \mathcal{A}_{\Delta_n} \\ f(X_1,\dots,X_i,\dots,X_n) &\longmapsto f(X_1,\dots,[\pi]_{\phi_i}(X_i),\dots,X_n) \end{aligned}$$

for $i \in \Delta_n$, where $\sigma_i \in \Gamma_{i,L}$ and $f \in \mathcal{A}_{\Delta_n}$. We also call the endomorphisms φ_i the *partial Frobenius* maps. The group action and the partial Frobenii commute with each other. To state the above more formally, let

$$\mathcal{T}_{+,\Delta_n,L} \coloneqq \prod_{i\in\Delta_n} \varphi_i^{\mathbb{N}_{\geq 0}}\Gamma_{i,L}$$

denote the commutative monoid generated by $\Gamma_{i,L}$ and the powers of φ_i. Then the above group action and partial Frobenii give us a semilinear action of $\mathcal{T}_{+,\Delta_n,L}$ on $\mathcal{A}_{\Delta_n}$.

We have that $\mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n} \simeq \mathcal{O}_{\Delta_n}/\pi\mathcal{O}_{\Delta_n} \simeq \kappa_L[\![X_1,\dots,X_n]\!][X_{\Delta_n}^{-1}]$, so we denote

$$E_{\Delta_n}^+ \coloneqq \kappa_L[\![X_1,\dots,X_n]\!]$$

and

$$E_{\Delta_n} \coloneqq E_{\Delta_n}^+[X_{\Delta_n}^{-1}].$$

Proposition 2.2. *(i) For all $i \in \Delta_n$, φ_i is injective on $\mathcal{A}_{\Delta_n}$.*

(ii) $\mathcal{A}_{\Delta_n}$ is a free $\varphi_i(\mathcal{A}_{\Delta_n})$-module with basis $1, X_i, \dots, X_i^{q-1}$.

Proof. (i) The map φ_i is $\mathcal{O}_L$-linear, therefore it descends to an endomorphism of $\mathcal{A}_{\Delta_n}/\pi^m\mathcal{A}_{\Delta_n}$ for all $m \in \mathbb{N}_{\geq 1}$. On $\mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n} \simeq E_{\Delta_n}$ the map φ_i becomes

$$\begin{aligned} \varphi_i : E_{\Delta_n} &\longrightarrow E_{\Delta_n} \\ f(X_1,\dots,X_i,\dots,X_n) &\longmapsto f(X_1,\dots,X_i^q,\dots,X_n) \end{aligned}$$

which is clearly injective. As π is not a zero divisor on $\mathcal{A}_{\Delta_n}$, we can inductively show that φ_i induces an injective map on $\mathcal{A}_{\Delta_n}/\pi^m\mathcal{A}_{\Delta_n}$ for every $m \geq 1$. By Lemma 2.1 we know that

$$\mathcal{A}_{\Delta_n} \simeq \varprojlim_m \mathcal{A}_{\Delta_n}/\pi^m\mathcal{A}_{\Delta_n}$$

and the conclusion follows.

(ii) Reformulating the statement, we want to show that $\mathcal{A}_{\Delta_n}$ with the $\mathcal{A}_{\Delta_n}$-module structure given by $a * b \coloneqq \varphi_i(a)b$ is isomorphic to the free $\mathcal{A}_{\Delta_n}$-module $\mathcal{A}_{\Delta_n}^{\oplus q}$, where the claimed isomorphism is the map

$$\begin{aligned} \mathcal{A}_{\Delta_n}^{\oplus q} &\longrightarrow \mathcal{A}_{\Delta_n} \\ (a_0,\dots,a_{q-1}) &\longmapsto \sum_{j=0}^{q-1} \varphi_i(a_j)X_i^j. \end{aligned}$$

Our maps are $\mathcal{O}_L$-linear and note that modulo π the map becomes

$$\begin{aligned} E_{\Delta_n}^{\oplus q} &\longrightarrow E_{\Delta_n} \\ (e_0,\dots,e_{q-1}) &\longmapsto \sum_{j=0}^{q-1} \varphi_i(e_j)X_i^j \end{aligned}$$

which is clearly bijective. From the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi^m\mathcal{A}_{\Delta_n}^{\oplus q}/\pi^{m+1}\mathcal{A}_{\Delta_n}^{\oplus q} \arrow[r] \arrow[d] & \mathcal{A}_{\Delta_n}^{\oplus q}/\pi^{m+1}\mathcal{A}_{\Delta_n}^{\oplus q} \arrow[r] \arrow[d] & \mathcal{A}_{\Delta_n}^{\oplus q}/\pi^{m}\mathcal{A}_{\Delta_n}^{\oplus q} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi^m\mathcal{A}_{\Delta_n}/\pi^{m+1}\mathcal{A}_{\Delta_n} \arrow[r] & \mathcal{A}_{\Delta_n}/\pi^{m+1}\mathcal{A}_{\Delta_n} \arrow[r] & \mathcal{A}_{\Delta_n}/\pi^{m}\mathcal{A}_{\Delta_n} \arrow[r] & 0 \end{tikzcd}$$

whose vertical arrows are the ones induced by our map $\mathcal{A}_{\Delta_n}^{\oplus q} \longrightarrow \mathcal{A}_{\Delta_n}$, it follows that the middle vertical arrow is bijective if the other two are. Using that $\pi^m \mathcal{A}_{\Delta_n}/\pi^{m+1}\mathcal{A}_{\Delta_n} \simeq \mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n}$, it follows inductively that the map

$$\begin{aligned} \mathcal{A}_{\Delta_n}^{\oplus q}/\pi^m \mathcal{A}_{\Delta_n}^{\oplus q} &\longrightarrow \mathcal{A}_{\Delta_n}/\pi^m \mathcal{A}_{\Delta_n} \\ (a_0, \dots, a_{q-1}) &\longmapsto \sum_{j=0}^{q-1} \varphi_i(a_j) X_i^j \end{aligned}$$

is bijective for every $m \in \mathbb{N}_{\geq 1}$. The result then follows by the π-adic completeness of $\mathcal{A}_{\Delta_n}$. $\square$

Given an $\mathcal{A}_{\Delta_n}$-module M and $i \in \Delta_n$, we form the new $\mathcal{A}_{\Delta_n}$-module $\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} M$ as an extension of scalars with the left factor $\mathcal{A}_{\Delta_n}$ considered as an $\mathcal{A}_{\Delta_n}$-module over itself through the endomorphism φ_i. This means that for $a, b \in \mathcal{A}_{\Delta_n}$ and $x \in M$, we have the relation $a\varphi_i(b) \otimes x = a \otimes bx$. An immediate consequence of Proposition 2.2 is the following.

Corollary 2.3. *The functor $\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} -$ from the category of $\mathcal{A}_{\Delta_n}$-modules into itself is exact.*

We let $\varphi \coloneqq \prod_{i \in \Delta_n} \varphi_i \in \mathcal{T}_{+,\Delta_n,L}$. Then $\varphi(a) \equiv a^q \bmod \pi\mathcal{A}_{\Delta_n}$ for every $a \in \mathcal{A}_{\Delta_n}$. Here are some more useful computations involving the action of φ on $\mathcal{A}_{\Delta_n}$.

Lemma 2.4. *For every $j \in \mathbb{N}$ we have that*

$$\varphi^{2j}(X_{\Delta_n}) \in X_{\Delta_n}^j \mathcal{O}_{\Delta_n}^+ + \pi^j \mathcal{O}_{\Delta_n}^+.$$

Proof. Lemma 1.7.7 (ii) in [Sch17] shows that

$$\varphi_i^{2j}(X_i) \in X_i^j \mathcal{O}_L[\![X_i]\!] + \pi^j X_i \mathcal{O}_L[\![X_i]\!]$$

for every $i \in \Delta_n$. Taking the product of these inclusions immediately proves the claim. $\square$

Lemma 2.5. *For $r, m \in \mathbb{N}_{>0}$, we have that*

$$\left(\frac{\varphi(X_{\Delta_n})}{X_{\Delta_n}}\right)^{rp^{m-1}} \in X_{\Delta_n}^{(q-1)rp^{m-1}} + \pi^m \mathcal{O}_{\Delta_n}^+.$$

Proof. Because $\varphi_i(X_i)$ is a Frobenius power series for π, we can write

$$\frac{\varphi_i(X_i)}{X_i} = X_i^{q-1} + \pi F_i(X_i)$$

for some $F_i(X_i) \in \mathcal{O}_L[\![X_i]\!]$. We deduce that

$$\begin{aligned}\left(\frac{\varphi_i(X_i)}{X_i}\right)^p &= \left(X_i^{q-1} + \pi F_i(X_i)\right)^p \\ &= X_i^{(q-1)p} + \sum_{j=1}^{p-1} \binom{p}{j} \pi^j F_i(X_i)^j X_i^{(q-1)(p-j)} + \pi^p F_i(X_i)^p \\ &= X_i^{(q-1)p} + \pi^2 Q_i(X_i)\end{aligned}$$

for some $Q_i(X_i) \in \mathcal{O}_L[\![X_i]\!]$ because $\pi | p$. Iterating this procedure, we obtain that

$$\left(\frac{\varphi_i(X_i)}{X_i}\right)^{p^{m-1}} = X_i^{(q-1)p^{m-1}} + \pi^m P_i(X_i)$$

for some $P_i(X_i) \in \mathcal{O}_L[\![X_i]\!]$. Therefore we obtain that

$$\begin{aligned}\left(\frac{\varphi(X_{\Delta_n})}{X_{\Delta_n}}\right)^{p^{m-1}} &= \prod_{i\in\Delta_n} \left(\frac{\varphi_i(X_i)}{X_i}\right)^{p^{m-1}} \\ &= \prod_{i\in\Delta_n} \left(X_i^{(q-1)p^{m-1}} + \pi^m P_i(X_i)\right) \\ &= X_{\Delta_n}^{(q-1)p^{m-1}} + \pi^m G(X_1, \dots X_n)\end{aligned}$$

for some $G(X_1, \dots X_n) \in \mathcal{O}^+_{\Delta_n}$. Raising this further to the r-th power, we obtain that

$$\begin{aligned}\left(\frac{\varphi(X_{\Delta_n})}{X_{\Delta_n}}\right)^{rp^{m-1}} &= \left(X_{\Delta_n}^{(q-1)p^{m-1}} + \pi^m G(X_1, \dots X_n)\right)^r \\ &= X_{\Delta_n}^{(q-1)rp^{m-1}} + \sum_{j=1}^{r} \binom{r}{j} \pi^{mj} G(X_1, \dots X_n)^j X_{\Delta_n}^{(q-1)p^{m-1}(r-j)} \\ &\in X_{\Delta_n}^{(q-1)rp^{m-1}} + \pi^m \mathcal{O}^+_{\Delta_n},\end{aligned}$$

as desired. □

Lemma 2.6. *Let $i \in \Delta_n$ and suppose that $\varphi_i(X_i) = X_i^q + \pi X_i F_i(X_i)$ for some $F_i(X_i) \in \mathcal{O}_L[\![X_i]\!]$. For $j > 0$ we have that*

$$\frac{1}{\varphi_i(X_i)} \in \sum_{k=0}^{j-1} (-1)^k \frac{\pi^k F_i(X_i)^k}{X_i^{(q-1)k+q}} + \pi^j \mathcal{A}_i.$$

Proof. A straightforward computation shows that

$$\begin{aligned}
\varphi_i(X_i)\left(\sum_{k=0}^{j-1}(-1)^k\frac{\pi^k F_i(X_i)^k}{X_i^{(q-1)k+q}}\right) &= (X_i^q+\pi X_i F_i(X_i))\left(\sum_{k=0}^{j-1}(-1)^k\frac{\pi^k F_i(X_i)^k}{X_i^{(q-1)k+q}}\right)\\
&= X_i^q\left(1+\frac{\pi F_i(X_i)}{X_i^{q-1}}\right)X_i^{-q}\left(\sum_{k=0}^{j-1}(-1)^k\frac{\pi^k F_i(X_i)^k}{X_i^{(q-1)k}}\right)\\
&= 1+(-1)^{j-1}\frac{\pi^j F_i(X_i)^j}{X_i^{(q-1)j}}\in 1+\pi^j\mathcal{A}_i,
\end{aligned}$$

therefore $\varphi_i(X_i)$ and $\displaystyle\sum_{k=0}^{j-1}(-1)^k\frac{\pi^k F_i(X_i)^k}{X_i^{(q-1)k+q}}$ revert each other in $\mathcal{A}_i/\pi^j\mathcal{A}_i$. □

2.2 Properties of E_{Δ_n}

In this section we explore some properties of the ring E_{Δ_n} and of certain finitely generated modules over E_{Δ_n}. For $n=1$, the ring E_{Δ_n} is identified with the field $\kappa_L(\!(X)\!)$. When $n\geq 2$, it is no longer true that E_{Δ_n} is a field and this is when our next results become relevant.

Lemma 2.7. *E_{Δ_n} is a Noetherian ring of finite global dimension.*

Proof. By Lemma 2.1 it follows that $E_{\Delta_n}\simeq\mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n}$ is a Noetherian ring. The ring $E_{\Delta_n}^+=\kappa_L[\![X_1,\dots,X_n]\!]$ is a regular local ring of Krull dimension n, in particular it is a regular ring. Therefore E_{Δ_n} is also regular, being a localization of a regular ring. The Krull dimension of E_{Δ_n} is finite, being a localization of a ring of finite Krull dimension. The global dimension of E_{Δ_n} is then finite as well because it agrees with its Krull dimension by regularity. □

Remark 2.8. One can further show that the Krull dimension $\dim E_{\Delta_n}$ of the ring E_{Δ_n} equals $n-1$. Indeed, since $E_{\Delta_n}^+$ is a local ring of dimension n and X_{Δ_n} belongs to the unique maximal ideal $(X_1,\dots,X_n)E_{\Delta_n}^+$ of $E_{\Delta_n}^+$, it follows that

$$\dim E_{\Delta_n}\leq n-1.$$

For the reverse inequality, we need to show that there exists a prime ideal in $\mathrm{Spec}(E_{\Delta_n}^+)$ of height $n-1$ which does not contain X_{Δ_n}. We start with the ideal $\mathfrak{p}=(X_1,\dots,X_{n-1})E_{\Delta_n}^+$ of height $n-1$ in $E_{\Delta_n}^+$. Consider the κ_L-linear ring automorphism of $E_{\Delta_n}^+$ that fixes X_n and maps X_i to X_i+X_n for all $1\leq i\leq n-1$. Then the image of $\mathfrak{p}$ under this automorphism

$$\mathfrak{q}=(X_1+X_n,\dots,X_{n-1}+X_n)E_{\Delta_n}^+$$

is a prime ideal of $E^+_{\Delta_n}$ of height $n-1$ as well. It does not contain X_n, else it would contain $X_1, \ldots, X_{n-1}$ as well, meaning that

$$(X_1, \ldots, X_n)E^+_{\Delta_n} \subseteq \mathfrak{q},$$

contradicting that $\operatorname{ht}\mathfrak{q} = n-1$ and $\operatorname{ht}(X_1, \ldots, X_n)E^+_{\Delta_n} = n$. It does not contain any X_i for $1 \le i \le n-1$, else it would also contain X_n, which was ruled out above. Therefore $X_{\Delta_n} \notin \mathfrak{q}$, hence $\mathfrak{q}$ is then our desired prime ideal.

Let us explore the action of $\Gamma_{\Delta_n,L}$ on E_{Δ_n} in more detail. The crucial lemma that will allow us to overcome the majority of the technical difficulties caused by working over the ring E_{Δ_n} for $n > 1$ is the following generalization of Lemma 2.1 in [Záb18a].

Lemma 2.9. *The only non-zero $\Gamma_{\Delta_n,L}$-invariant ideal of E_{Δ_n} is E_{Δ_n}.*

Proof. Step 1: We reduce to the case when $\phi_n(t) = \pi t + t^q$. We let $\phi(t) = \pi t + t^q$ and consider the κ_L-linear ring homomorphism

$$\rho : E_{\Delta_n} \longrightarrow E_{\Delta_n}$$

mapping X_n to $\overline{[1]}_{\phi_n,\phi}(X_n)$ and leaving the other variables unchanged (the element 1 can be replaced by any unit in $\mathcal{O}_L$). Then ρ is $\Gamma_{\Delta_n,L}$-equivariant if on the left hand side E_{Δ_n} is equipped with our previously defined $\Gamma_{\Delta_n,L}$-action, while on the right hand side $\sigma \in \Gamma_{n,L}$ maps X_n to $\overline{[\chi_L(\sigma)]}_{\phi}(X_n)$ instead and leaves the other variables unchanged, while the action of the other components of $\Gamma_{\Delta_n,L}$ remains the same as the previously defined one. Indeed, if $\chi_L(\sigma) = a \in \mathcal{O}_L$ for $\sigma \in \Gamma_{n,L}$, then

$$\begin{aligned}
\rho(\sigma \cdot X_n) &= \rho(\overline{[a]}_{\phi_n}(X_n)) = \overline{[a]}_{\phi_n}(\overline{[1]}_{\phi_n,\phi}(X_n)) \\
&= \overline{[a]}_{\phi_n,\phi}(X_n) = \overline{[1]}_{\phi_n,\phi}(\overline{[a]}_{\phi}(X_n)) \\
&= \sigma \cdot \rho(X_n)
\end{aligned}$$

by Lemma 1.25. From Lemma 1.25 we also know that

$$[1]_{\phi,\phi_n}([1]_{\phi_n,\phi}(t)) = [1]_{\phi}(t) = t = [1]_{\phi_n}(t) = [1]_{\phi_n,\phi}([1]_{\phi,\phi_n}(t)).$$

Therefore ρ is an automorphism of E_{Δ_n} which preserves the nonzero $\Gamma_{\Delta_n,L}$-invariant ideals on both sides. Hence from now on we may assume that ϕ_n is equal to ϕ.

Step 2: Let $\zeta_\ell \coloneqq 1 + \pi^\ell$ for $\ell \in \mathbb{N}_{\ge 1}$. Write

$$[\zeta_\ell]_{\phi}(t) = \zeta_\ell t + \sum_{i=2}^{\infty} b_{i,\ell} t^i$$

for $b_{i,\ell} \in \mathcal{O}_L$. By Lemma 0.1 of [GK20] we know that $b_{i,\ell} = 0$ whenever $i-1 \notin (q-1)\mathbb{N}$. By Proposition 1.24 we have the relation $\phi([\zeta_\ell]_\phi(t)) = [\zeta_\ell]_\phi(\phi(t))$, therefore

$$\pi\left(\zeta_\ell t + \sum_{i\in(q-1)\mathbb{N}+1} b_{i,\ell} t^i\right) + \left(\zeta_\ell t + \sum_{i\in(q-1)\mathbb{N}+1} b_{i,\ell} t^i\right)^q = \zeta_\ell(\pi t + t^q) + \sum_{i\in(q-1)\mathbb{N}+1} b_{i,\ell}(\pi t + t^q)^i. \tag{2.1}$$

We claim that

$$\overline{[\zeta_\ell]}_\phi(t) \in t + t^{q^\ell}\kappa_L[\![t^{q-1}]\!]^\times. \tag{2.2}$$

For that, more generally we show that

- $b_{q^r,\ell} \in \pi^{\ell-r}\mathcal{O}_L^\times$ for $1 \le r \le \ell$, and (2.3)
- $b_{k(q-1)+1,\ell} \in \pi^{\ell-r}\mathcal{O}_L$ for $\dfrac{q^r-1}{q-1} \le k < \dfrac{q^{r+1}-1}{q-1}$ and $1 \le r \le \ell-1$. (2.4)

We do so by iterating r. For the case $r=1$, by checking the coefficient of t^q in both sides of (2.1) we obtain that

$$b_{q,\ell} = \frac{\zeta_\ell^q - \zeta_\ell}{\pi^q - \pi} = \frac{(\zeta_\ell - 1)(\zeta_\ell + \zeta_\ell^2 + \ldots + \zeta_\ell^{q-1})}{\pi(\pi-1)(1+\pi+\ldots\pi^{q-2})} = \frac{\pi^{\ell-1}(\zeta_\ell + \zeta_\ell^2 + \ldots + \zeta_\ell^{q-1})}{(\pi-1)(1+\pi+\ldots\pi^{q-2})} \in \pi^{\ell-1}\mathcal{O}_L^\times$$

where we used that $\zeta_\ell + \zeta_\ell^2 + \ldots + \zeta_\ell^{q-1} \equiv q-1 \bmod \pi\mathcal{O}_L$. Thus (2.3) is proven for $r=1$.

To check (2.4) for $r=1$, we suppose that $\ell \ge 2$, else the condition $1 \le r \le \ell-1$ is not satisfied. By (2.3) for $r=1$, we already showed that (2.4) holds for $r=k=1$, so let $2 \le k < q+1$. Suppose that we have shown that $b_{k_0(q-1)+1,\ell} \in \pi^{\ell-1}\mathcal{O}_L$ for all $1 \le k_0 < k$. We compute the coefficient of $t^{k(q-1)+1}$ in both sides of (2.1).

For the left hand side, the coefficient of $t^{k(q-1)+1}$ in $\pi\left(\zeta_\ell t + \sum_{i\in(q-1)\mathbb{N}+1} b_{i,\ell}t^i\right)$ equals $\pi b_{k(q-1)+1,\ell}$. Also, we claim that the desired coefficient in

$$\left(\zeta_\ell t + b_{q,\ell}t^q + \ldots + b_{(k-1)(q-1)+1,\ell}t^{(k-1)(q-1)+1}\right)^q$$

equals $q\zeta_\ell^{q-1}b_{(k-1)(q-1)+1,\ell} + \pi^\ell c$ for some $c \in \mathcal{O}_L$. Indeed, to obtain a term with the desired power of t in

$$\left(\zeta_\ell t + b_{q,\ell}t^q + \ldots + b_{(k-1)(q-1)+1,\ell}t^{(k-1)(q-1)+1}\right)^q,$$

one needs to either factor at least two terms with a power of t larger than 1 or take

$$(\zeta_\ell t)^{q-1}b_{(k-1)(q-1)+1,\ell}t^{(k-1)(q-1)+1}.$$

In the first case we obtain a coefficient term divisible by $\pi^{2(\ell-1)}$ by the induction hypothesis on k, and thus divisible by π^ℓ since $\ell \ge 2$. In the second case, from the multinomial expansion we obtain the coefficient term $q\zeta_\ell^{q-1}b_{(k-1)(q-1)+1,\ell}$. Because $\pi^{\ell-1}|b_{(k-1)(q-1)+1,\ell}$ by our induction hypothesis for k and since $\pi|q$, for the desired coefficient we can also write

$$q\zeta_\ell^{q-1}b_{(k-1)(q-1)+1,\ell} + \pi^\ell c = \pi^\ell c'$$

for some $c' \in \mathcal{O}_L$.

For the right hand side, the desired coefficient in $b_{j(q-1)+1,\ell}(\pi t + t^q)^{j(q-1)+1}$ equals

$$\binom{j(q-1)+1}{k-j} b_{j(q-1)+1,\ell}\pi^{jq+1-k}$$

when $1 \leq j \leq k$ and 0 otherwise. Overall the desired coefficient in the right hand side of (2.1) equals

$$\begin{aligned}
&\sum_{j=1}^{k} \binom{j(q-1)+1}{k-j} b_{j(q-1)+1,\ell}\pi^{jq+1-k} \\
&= \sum_{j=1}^{k-1} \binom{j(q-1)+1}{k-j} b_{j(q-1)+1,\ell}\pi^{jq+1-k} + b_{k(q-1)+1,\ell}\pi^{k(q-1)+1} \\
&= b_{k(q-1)+1,\ell}\pi^{k(q-1)+1} + \pi^{\ell} c''
\end{aligned}$$

for some $c'' \in \mathcal{O}_L$. In the last equality we used that $jq+1-k \geq q+1-k \geq 1$ and $b_{j(q-1)+1,\ell} \in \pi^{\ell-1}\mathcal{O}_L$ for $1 \leq j < k$ by the induction hypothesis. Therefore

$$b_{k(q-1)+1,\ell} = \frac{\pi^{\ell}c'' - \pi^{\ell}c'}{\pi - \pi^{k(q-1)+1}} \in \pi^{\ell-1}\mathcal{O}_L$$

and thus (2.4) holds for $r = 1$.

Suppose that $r > 1$ and that (2.3) and (2.4) hold for all $1 \leq r_0 < r$. We now show that these hold for r as well. We start by showing that $b_{q^r,\ell} \in \pi^{\ell-r}\mathcal{O}_L^{\times}$, for which we compute the coefficient of t^{q^r} in both sides of (2.1).

For the left hand side, the coefficient of t^{q^r} in $\pi\left(\zeta_\ell t + \sum_{i\in(q-1)\mathbb{N}+1} b_{i,\ell}t^i\right)$ equals $\pi b_{q^r,\ell}$.

We now compute the coefficient of t^{q^r} in $\left(\zeta_\ell t + b_{q,\ell}t^q + \ldots + b_{q^r-(q-1),\ell}t^{q^r-(q-1)}\right)^q$. We claim the desired coefficient in the last expression equals

$$q\zeta_\ell^{q-1} b_{q^r-(q-1),\ell} + \pi^{\ell-r+2}d$$

for some $d \in \mathcal{O}_L$. Indeed, to obtain a term with the desired power of t in

$$\left(\zeta_\ell t + b_{q,\ell}t^q + \ldots + b_{q^r-(q-1),\ell}t^{q^r-(q-1)}\right)^q,$$

one needs to either factor at least two terms with a power of t larger than 1 or take $(\zeta_\ell t)^{q-1} b_{q^r-(q-1),\ell}t^{q^r-(q-1)}$. In the first case, by the induction hypothesis we obtain a coefficient term divisible by $\pi^{2(\ell-r+1)}$ and thus divisible by $\pi^{\ell-r+2}$ since $\ell \geq r$. In the second case, from the multinomial expansion we obtain the coefficient term $q\zeta_\ell^{q-1} b_{q^r-(q-1),\ell}$. Because $\pi^{\ell-r+1} | b_{q^r-(q-1),\ell}$ by the induction hypothesis and since $\pi | q$, for the desired coefficient we can also write

$$q\zeta_\ell^{q-1} b_{q^r-(q-1),\ell} + \pi^{\ell-r+2}d = \pi^{\ell-r+2}d'$$

for some $d' \in \mathcal{O}_L$.

For the right hand side, the desired coefficient in $b_{j(q-1)+1,\ell}(\pi t + t^q)^{j(q-1)+1}$ equals

$$\binom{j(q-1)+1}{\frac{q^r-1}{q-1}-j} b_{j(q-1)+1,\ell}\pi^{jq+1-\frac{q^r-1}{q-1}}$$

when $\frac{q^{r-1}-1}{q-1} \le j \le \frac{q^r-1}{q-1}$ and 0 otherwise. Therefore the desired coefficient in the right hand side of (2.1) equals

$$\begin{aligned}
&\sum_{j=\frac{q^{r-1}-1}{q-1}}^{\frac{q^r-1}{q-1}} \binom{j(q-1)+1}{\frac{q^r-1}{q-1}-j} b_{j(q-1)+1,\ell}\pi^{jq+1-\frac{q^r-1}{q-1}} = \\
&= b_{q^{r-1},\ell} + \sum_{j=\frac{q^{r-1}-1}{q-1}+1}^{\frac{q^r-1}{q-1}-1} \binom{j(q-1)+1}{\frac{q^r-1}{q-1}-j} b_{j(q-1)+1,\ell}\pi^{jq+1-\frac{q^r-1}{q-1}} + b_{q^r,\ell}\pi^{q^r} \\
&= b_{q^{r-1},\ell} + \pi^{\ell-r+2}d'' + b_{q^r,\ell}\pi^{q^r}
\end{aligned}$$

for some $d'' \in \mathcal{O}_L$. In the last equality, we used that $jq+1-\frac{q^r-1}{q-1} \ge 1$ and $b_{j(q-1)+1,\ell} \in \pi^{\ell-r+1}\mathcal{O}_L$ for $\frac{q^{r-1}-1}{q-1} < j < \frac{q^r-1}{q-1}$ by the induction hypothesis of (2.4) for $r-1$. We then obtain

$$b_{q^r,\ell} = \frac{b_{q^{r-1},\ell} + \pi^{\ell-r+2}d'' - \pi^{\ell-r+2}d'}{\pi - \pi^{q^r}} \in \pi^{\ell-r}\mathcal{O}_L^\times$$

from the assumption that $b_{q^{r-1},\ell} \in \pi^{\ell-r+1}\mathcal{O}_L^\times$ and (2.3) is proven.

Now, let us show that (2.4) holds. If $\ell = r$, then the inequality in the condition of (2.4) is not satisfied and there is nothing to prove, so suppose that $\ell > r$. By the above, we already showed that (2.4) holds for $k = \frac{q^r-1}{q-1}$ so let $\frac{q^r-1}{q-1} < k < \frac{q^{r+1}-1}{q-1}$. Suppose that we have shown that $b_{k_0(q-1)+1,\ell} \in \pi^{\ell-r}\mathcal{O}_L$ for all $\frac{q^r-1}{q-1} \le k_0 < k$. Again, we compute the coefficient of $t^{k(q-1)+1}$ in both sides of (2.1).

For the left hand side, the coefficient of $t^{k(q-1)+1}$ in $\pi\left(\zeta_\ell t + \sum_{i\in(q-1)\mathbb{N}+1} b_{i,\ell}t^i\right)$ equals $\pi b_{k(q-1)+1,\ell}$. We now compute the coefficient of $t^{k(q-1)+1}$ in

$$\left(\zeta_\ell t + b_{q,\ell}t^q + \ldots + b_{(k-1)(q-1)+1,\ell}t^{(k-1)(q-1)+1}\right)^q.$$

We claim the desired coefficient in the last expression equals

$$q\zeta_\ell^{q-1}b_{(k-1)(q-1)+1,\ell} + \pi^{\ell-r+1}c$$

for some $c \in \mathcal{O}_L$. Indeed, to obtain a term with the desired power of t in

$$\left(\zeta_\ell t + b_{q,\ell}t^q + \ldots + b_{(k-1)(q-1)+1,\ell}t^{(k-1)(q-1)+1}\right)^q,$$

one needs to either factor at least two terms with a power of t larger than 1 or take

$$(\zeta_\ell t)^{q-1} b_{(k-1)(q-1)+1,\ell} t^{(k-1)(q-1)+1}.$$

In the first case we obtain a term divisible by $\pi^{2(\ell-r)}$ and thus divisible by $\pi^{\ell-r+1}$ since $\ell > r$. In the second case, from the multinomial expansion we obtain the coefficient term $q\zeta_\ell^{q-1} b_{(k-1)(q-1)+1,\ell}$. Because $\pi^{\ell-r} | b_{(k-1)(q-1)+1,\ell}$ by the induction hypothesis for k and since $\pi | q$, for the desired coefficient we can also write

$$q\zeta_\ell^{q-1} b_{(k-1)(q-1)+1,\ell} + \pi^{\ell-r+1} c = \pi^{\ell-r+1} c'$$

for some $c' \in \mathcal{O}_L$.

For the right hand side, the desired coefficient in $b_{j(q-1)+1,\ell}(\pi t + t^q)^{j(q-1)+1}$ equals

$$\binom{j(q-1)+1}{k-j} b_{j(q-1)+1,\ell} \pi^{jq+1-k}$$

when $\lceil \frac{k-1}{q} \rceil \leq j \leq k$ and 0 otherwise. Therefore the desired coefficient in the right hand side of (2.1) equals

$$\begin{aligned} &\sum_{j=\lceil \frac{k-1}{q} \rceil}^{k} \binom{j(q-1)+1}{k-j} b_{j(q-1)+1,\ell} \pi^{jq+1-k} \\ &= \sum_{j=\lceil \frac{k-1}{q} \rceil}^{k-1} \binom{j(q-1)+1}{k-j} b_{j(q-1)+1,\ell} \pi^{jq+1-k} + b_{k(q-1)+1,\ell} \pi^{k(q-1)+1}. \end{aligned}$$

We claim that

$$\sum_{j=\lceil \frac{k-1}{q} \rceil}^{k-1} \binom{j(q-1)+1}{k-j} b_{j(q-1)+1,\ell} \pi^{jq+1-k} = \pi^{\ell-r+1} c''$$

for some $c'' \in \mathcal{O}_L$. Indeed, in the above sum we have that $jq+1-k > 0$ unless $q | (k-1)$ and

$$j = \left\lceil \frac{k-1}{q} \right\rceil = \frac{k-1}{q}.$$

When $jq+1-k > 0$ we have that $\pi | \pi^{jq+1-k}$, while $\pi^{\ell-r} | b_{j(q-1)+1,\ell}$ by the induction hypothesis. When $\frac{k-1}{q} \in \mathbb{N}$ note that

$$\frac{k-1}{q} < \frac{(q^r + \ldots + q + 1) - 1}{q} = \frac{q^{r-1} - 1}{q-1}$$

therefore by the induction hypothesis of (2.4) for $r-1$ we have that

$$b_{\frac{k-1}{q}(q-1)+1,\ell} \in \pi^{\ell-r+1} \mathcal{O}_L$$

and our claim follows. Therefore

$$b_{k(q-1)+1,\ell} = \frac{\pi^{\ell-r+1}c'' - \pi^{\ell-r+1}c'}{\pi - \pi^{k(q-1)+1}} \in \pi^{\ell-r}\mathcal{O}_L$$

and thus (2.4) is proven.

Step 3: We generalize (2.2) to arbitrary powers of t. We claim that

$$\zeta_\ell \cdot t^{up^e} \in t^{up^e} + t^{(u+q^\ell-1)p^e}\kappa_L[\![t]\!]^\times \tag{2.5}$$

where $e \in \mathbb{N}$, $u \in \mathbb{Z}$ is coprime to p and $\zeta_\ell \cdot t^{up^e}$ denotes the action of ζ_ℓ on t^{up^e}. By (2.2) we have that

$$\overline{[\zeta_\ell]}_\phi(t) \in t + \bar{b}_{q^\ell,\ell}t^{q^\ell} + t^{q^\ell+q-1}\kappa_L[\![t]\!]$$

where $\bar{b}_{q^\ell,\ell} := b_{q^\ell,\ell} \bmod \pi\mathcal{O}_L$. Therefore

$$\zeta_\ell \cdot t^u = (t + \bar{b}_{q^\ell,\ell}t^{q^\ell} + \ldots)^u \in t^u + u\bar{b}_{q^\ell,\ell}t^{u+q^\ell-1}\kappa_L[\![t]\!]^\times = t^u + t^{u+q^\ell-1}\kappa_L[\![t]\!]^\times$$

where the last equality holds by our assumption about u and because $\bar{b}_{q^\ell,\ell} \in \kappa_L^\times$ by (2.3). Raising this further to the power of p^e we obtain (2.5), as desired.

Step 4: The rest of the proof is analogous to the one of Lemma 2.2 in [Záb18a]. Let I be a non-zero ideal in E_{Δ_n} stable under the action of $\Gamma_{\Delta_n,L}$. We show that $I = E_{\Delta_n}$ by induction on the number of variables. When $n = 1$, the ring E_{Δ_n} is a field and the statement becomes trivial, so assume that $n \geq 2$.

Let $J = I \cap E^+_{\Delta_n}$. Multiplying a nonzero element of I by a suitable monomial, we obtain that J is a nonzero ideal of $E^+_{\Delta_n}$. Furthermore, by Krull's intersection theorem the ideal J is a closed subset of $E^+_{\Delta_n}$, if we equip the latter with the $(X_1, \ldots, X_n)$-adic topology. Let

$$J_0 := \{\widetilde{g} \in E^+_{\Delta_{n-1}} : \exists g = \sum_{j=0}^{\infty} g_j X_n^j \in J \text{ with } g_j \in E^+_{\Delta_{n-1}} \text{ such that } g_0 = \widetilde{g}\}.$$

We now show that there is an element

$$h = \sum_{j=0}^{\infty} h_j X_n^j$$

of J where $h_j \in E^+_{\Delta_{n-1}}$, such that $h_0 \neq 0$ and for every $j > 0$ we either have $h_j = 0$ or $h_j \notin J_0$. We start with an arbitrary element

$$g = \sum_{j=0}^{\infty} g_j X_n^j$$

of J where $g_j \in E^+_{\Delta_{n-1}}$ and $g_0 \neq 0$. If g satisfies the desired property for h then we are done. Else, suppose that $k > 0$ is the smallest positive index for which $g_k \in J_0 - \{0\}$. Consider

$$g' = \sum_{j=0}^{\infty} g'_j X_n^j \in J$$

where $g'_j \in E^+_{\Delta_{n-1}}$, with $g'_0 = g_k$ and take $g'' = g - X_n^k g' \in J$. One can continue eliminating the positive powers of X_n whose coefficients lie in $J_0 - \{0\}$ and use that the ideal J is closed for the $(X_1, \dots, X_n)E^+_{\Delta_n}$-adic topology to obtain the desired element h.

If $h_j = 0$ for all $j > 0$, then $h = h_0 \in E^+_{\Delta_{n-1}} - \{0\}$. Therefore $I \cap E_{\Delta_{n-1}}$ is a nonzero $\Gamma_{\Delta_{n-1},L}$-invariant ideal of $E_{\Delta_{n-1}}$. Then by the induction hypothesis $1 \in I \cap E_{\Delta_{n-1}} \subseteq I$ and we are done.

Else, consider the non-empty set $\mathcal{S} = \{j \in \mathbb{N}_{\geq 1} : h_j \neq 0\}$. For $j \in \mathcal{S}$ write

$$j = u(j)p^{r(j)}$$

where $u(j)$ and $r(j)$ are non-negative integers such that $u(j)$ and p are coprime. Consider the following total order on the elements of $\mathcal{S}$: for $j = u(j)p^{r(j)}$ and $j' = u(j')p^{r(j')}$ in $\mathcal{S}$ we write $j \prec j'$ if $r(j) < r(j')$ or if $r(j) = r(j')$ and $u(j) < u(j')$. Let j_0 be the smallest element of $\mathcal{S}$ with respect to $\prec$.

For $\ell > 0$ let $\theta_{\ell,n}$ be the κ_L-linear ring automorphism of E_{Δ_n} that maps X_n to $\overline{[\zeta_\ell]}_\phi(X_n)$ and leaves X_i unchanged for $i \in \Delta_{n-1}$. Consider the element

$$\theta_{\ell,n}(h) - h = \sum_{j \in \mathcal{S}} h_j(\theta_{\ell,n}(X_n^j) - X_n^j) \in J.$$

For $j = u(j)p^{r(j)} \in \mathcal{S}$ by formula (2.5) we have that

$$\mathrm{ord}_{X_n}(\theta_{\ell,n}(X_n^j) - X_n^j) = \left(u(j) + q^\ell - 1\right) p^{r(j)}.$$

We now choose an appropriate ℓ such that

$$\left(u(j) + q^\ell - 1\right) p^{r(j)} > \left(u(j_0) + q^\ell - 1\right) p^{r(j_0)} \tag{2.6}$$

holds for all $j \in \mathcal{S} - \{j_0\}$. We claim that for $\ell > 0$ large enough such that

$$q^\ell - 1 > \frac{u(j_0)}{p-1}$$

is true, the inequality (2.6) holds. Indeed, note that for $j \in \mathcal{S} - \{j_0\}$ we either have that $r(j_0) < r(j)$ or that $r(j_0) = r(j)$ and $u(j_0) < u(j)$. If $r(j_0) = r(j)$ and $u(j_0) < u(j)$ the inequality (2.6) is clear. If $r(j_0) < r(j)$ then

$$\begin{aligned}
\left(u(j) + q^\ell - 1\right) p^{r(j)} &> \left(q^\ell - 1\right) p^{r(j)} \geq (q^\ell - 1)p^{r(j_0)+1} \\
&= (q^\ell - 1)(p-1)p^{r(j_0)} + (q^\ell - 1)p^{r(j_0)} \\
&> \left(u(j_0) + q^\ell - 1\right) p^{r(j_0)}.
\end{aligned}$$

Therefore

$$\mathrm{ord}_{X_n}(\theta_{\ell,n}(h) - h) = \mathrm{ord}_{X_n}(\theta_{\ell,n}(X_n^{j_0}) - X_n^{j_0}) = \left(u(j_0) + q^\ell - 1\right) p^{r(j_0)}$$

and in $X_n^{-(u(j_0)+q^\ell-1)p^{r(j_0)}}(\theta_{\ell,n}(h) - h) \in J$ the terms for which the power of X_n is zero add up to a $\kappa_L^\times$-multiple of h_{j_0}, contradicting that $h_{j_0} \notin J_0$. $\square$

A first application of Lemma 2.9 will be to show that finitely generated modules over E_{Δ_n} admitting a semilinear $\Gamma_{\Delta_n,L}$-action are projective. For proving this we need one more result that shows that E_{Δ_n} regarded as a module over itself, admits an injective resolution whose objects admit a semilinear $\Gamma_{\Delta_n,L}$-action and whose morphisms are $\Gamma_{\Delta_n,L}$-equivariant. The proof is identical to the one of Lemma 2.1 in [Záb18b].

Lemma 2.10. *There exists an injective resolution of E_{Δ_n} in $E_{\Delta_n} - \mathrm{Mod}$*

$$0 \to E_{\Delta_n} \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \dots$$

such that I^i admits a semilinear $\Gamma_{\Delta_n,L}$-action for each $i \geq 0$ and every morphism is $\Gamma_{\Delta_n,L}$-equivariant. Moreover, the resolution can be chosen so that I^i is a torsion module for all $i \geq 1$.

Proof. More generally, consider the Cousin complex of R-modules associated to an arbitrary commutative Noetherian ring R defined as follows.

For each $i \geq 0$ let $\mathcal{H}_i(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathrm{ht}(\mathfrak{p}) = i\}$. Let $I^0(R) \coloneqq \bigoplus_{\mathfrak{p}\in\mathcal{H}_0(R)} R_{\mathfrak{p}}$ and $d^{-1} : R \to I^0(R)$ be the map defined by

$$d^{-1}(r) = \left(\left(\frac{r}{1}\right)_{\mathfrak{p}}\right)_{\mathfrak{p}\in\mathcal{H}_0(R)}$$

where $\left(\frac{r}{1}\right)_{\mathfrak{p}}$ denotes the image of $r \in R$ in the localized ring $R_{\mathfrak{p}}$. The map d^{-1} is well defined because a Noetherian ring has finitely many minimal prime ideals, which means that the set $\mathcal{H}_0(R)$ is finite.

For $i \geq 1$, define recursively $I^i(R) \coloneqq \bigoplus_{\mathfrak{p}\in\mathcal{H}_i(R)} (\mathrm{coker}\, d^{i-2})_{\mathfrak{p}}$ and $I^{i-1}(R) \xrightarrow{d^{i-1}} I^i(R)$ to be the composition of

$$I^{i-1}(R) \to \mathrm{coker}\, d^{i-2} \to \bigoplus_{\mathfrak{p}\in\mathcal{H}_i} (\mathrm{coker}\, d^{i-2})_{\mathfrak{p}}$$

where the first arrow is the natural projection map, and the second one maps $x \in \mathrm{coker}\, d_{i-2}$ to $\left(\left(\frac{x}{1}\right)_{\mathfrak{p}}\right)_{\mathfrak{p}\in\mathcal{H}_i}$. By a repeated application of Proposition 2.5 (iii) of [Sha69] we obtain that

$$\mathrm{Supp}(\mathrm{coker}\, d_{i-2}) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathrm{ht}(\mathfrak{p}) \geq i\}$$

and Proposition 2.2 of [Sha69] implies that the second arrow is well defined.

In the case when $R = E_{\Delta_n}$, let $I^i \coloneqq I^i(E_{\Delta_n})$. Since each element of $\Gamma_{\Delta_n,L}$ induces an automorphism on E_{Δ_n} leaving invariant the height of the primes, it is easy to check that $\Gamma_{\Delta_n,L}$ acts on each I^i semilinearly and that the differential maps are

$\Gamma_{\Delta_n,L}$-equivariant. By Theorem 5.4 of [Sha69] the Cousin complex of a Noetherian ring is an injective resolution if the ring is Gorenstein. By Lemma 2.7 E_{Δ_n} is such, since it is regular.

For the last statement of the lemma one can more generally show that $I^i(R)$ is a torsion module for any Gorenstein integral domain R for all $i \geq 1$. By Corollary 5.5 of [Sha69]

$$(\operatorname{coker} d^{i-2})_{\mathfrak{p}} \simeq E_R(R/\mathfrak{p})$$

for all $\mathfrak{p} \in \mathcal{H}_i(R)$ and $i \geq 0$, where by convention d^{-2} is the arrow $0 \to R$ in our complex and $E_R(R/\mathfrak{p})$ denotes the injective hull in $R-$Mod of the module $R/\mathfrak{p}$. Therefore

$$I^i(R) = \bigoplus_{\mathfrak{p} \in \mathcal{H}_i(R)} E_R(R/\mathfrak{p})$$

for all $i \geq 0$. Because R is an integral domain, it suffices to show that $E_R(R/\mathfrak{p})$ is a torsion R-module for each $\mathfrak{p} \in \mathcal{H}_i(R)$ for all $i \geq 1$.

Since $E_R(R/\mathfrak{p})$ is an injective hull of $R/\mathfrak{p}$ in R-Mod, it means that $R/\mathfrak{p}$ may be identified with an essential submodule of $E_R(R/\mathfrak{p})$. Let x be a non-zero element of $E_R(R/\mathfrak{p})$. Then Rx is a nonzero R-submodule of $E_R(R/\mathfrak{p})$ therefore $(R/\mathfrak{p}) \cap Rx \neq 0$. This implies that $E_R(R/\mathfrak{p})/(R/\mathfrak{p})$ is a torsion R-module. Since $R/\mathfrak{p}$ is a torsion R-module when $\operatorname{ht}(\mathfrak{p}) > 0$ the conclusion follows because R is an integral domain. □

We can finally show the projectivity of finitely generated modules over E_{Δ_n} admitting a semilinear $\Gamma_{\Delta_n,L}$-action exactly as in the proof of Lemma 2.2 of [Záb18b].

Lemma 2.11. *Suppose that D is a finitely generated module over E_{Δ_n} equipped with a semilinear action of $\Gamma_{\Delta_n,L}$. Then D is a projective E_{Δ_n}-module.*

Proof. The ring E_{Δ_n} has finite global dimension by Lemma 2.7, therefore D has finite projective dimension m. Suppose that D is not projective, meaning that $m > 0$. By Lemma 4.16 of [Wei94], we know that $\operatorname{Ext}^i_{E_{\Delta_n}}(D, M) = 0$ for all $i > m$ and all E_{Δ_n}-modules M and that there exists an E_{Δ_n}-module M_0 such that $\operatorname{Ext}^m_{E_{\Delta_n}}(D, M_0) \neq 0$. Consider a short exact sequence of E_{Δ_n}-modules

$$0 \to K \to E_{\Delta_n}^{\oplus S} \to M_0 \to 0.$$

Then by the long exact sequence of Ext it follows that we have an exact sequence

$$\operatorname{Ext}^m_{E_{\Delta_n}}(D, K) \to \operatorname{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n}^{\oplus S}) \to \operatorname{Ext}^m_{E_{\Delta_n}}(D, M_0) \to 0$$

and therefore $\operatorname{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n}^{\oplus S}) \neq 0$. Since E_{Δ_n} is a Noetherian ring and D is a finitely generated E_{Δ_n}-module, the functor $\operatorname{Ext}^m_{E_{\Delta_n}}(D, -)$ commutes with filtered colimits and it follows that

$$\operatorname{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n}) \neq 0. \tag{2.7}$$

We now compute $\mathrm{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n})$ using the injective resolution

$$0 \to E_{\Delta_n} \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \dots$$

of E_{Δ_n} from Lemma 2.10 where I^i is a torsion E_{Δ_n}-module for every $i \geq 1$. Considering the induced complex

$$0 \to \mathrm{Hom}_{E_{\Delta_n}}(D, E_{\Delta_n}) \xrightarrow{d_*^{-1}} \mathrm{Hom}_{E_{\Delta_n}}(D, I^0) \xrightarrow{d_*^0} \mathrm{Hom}_{E_{\Delta_n}}(D, I^1) \xrightarrow{d_*^1} \dots$$

we have that $\mathrm{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n}) \simeq \ker d_*^m / \operatorname{im} d_*^{m-1}$. Both D and I^i admit a semilinear $\Gamma_{\Delta_n,L}$-action, therefore so does $\mathrm{Hom}_{E_{\Delta_n}}(D, I^i)$ for all $i \geq 0$ if we define

$$(\sigma\varphi)(y) \coloneqq \sigma(\varphi(\sigma^{-1}(y))),$$

for all $\varphi \in \mathrm{Hom}_{E_{\Delta_n}}(D, I^i), \sigma \in \Gamma_{\Delta_n,L}$ and $y \in D$. Moreover, this action makes d_*^i a $\Gamma_{\Delta_n,L}$-equivariant map for all $i \geq 0$. In particular $\Gamma_{\Delta_n,L}$ acts semilinearly on $\mathrm{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n})$.

Since D is finitely generated and E_{Δ_n} is a domain, it follows that every element of $\mathrm{Hom}_{E_{\Delta_n}}(D, I^m)$ is torsion since I^m is a torsion module. Therefore $\mathrm{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n})$ is a torsion module as well. Since D is finitely generated and E_{Δ_n} is Noetherian, using a resolution of D consisting of finitely generated free E_{Δ_n}-modules shows that $\mathrm{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n})$ is finitely generated over E_{Δ_n}. Since $\mathrm{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n})$ is a torsion module it has a nonzero global annihilator ideal J. But J must also be a $\Gamma_{\Delta_n,L}$-invariant ideal of E_{Δ_n}. By Lemma 2.9 it follows that $J = E_{\Delta_n}$ and thus $\mathrm{Ext}^m_{E_{\Delta_n}}(D, E_{\Delta_n}) = 0$, contradicting (2.7). □

In Lemma 2.3 of [Záb18b] it is shown that a finitely generated projective module over $\mathbb{F}_p[\![X_1, \dots, X_n]\!][X_{\Delta_n}^{-1}]$ is automatically stably free. The proof remains the same if we replace $\mathbb{F}_p$ with κ_L.

Lemma 2.12. *Every finitely generated projective module over E_{Δ_n} is stably free.*

2.3 The topologies

In the one variable case the ring E_L is equipped with the X-adic topology, that is the topology for which the sets $X^\ell \kappa_L[\![X]\!]$ form a fundamental system of open neighbourhoods of zero for $\ell \in \mathbb{Z}$. The lift of this topology to $\mathcal{A}_L$, for which the sets $X^\ell \mathcal{O}_L[\![X]\!] + \pi^m \mathcal{A}_L$ form a fundamental system of open neighbourhoods of zero for $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$, is the topology considered in [Sch17] on the objects of the relevant category of modules. We will generalize this topology in two ways in the multivariable case and we will later see what purposes each of them serve.

Our first generalization of the X-adic topology in the multivariable case is the topology on E_{Δ_n} for which the sets $y + X_{\Delta_n}^\ell E_{\Delta_n}^+$ form a fundamental system of open neighbourhoods of $y \in E_{\Delta_n}$ for $\ell \in \mathbb{Z}$. We call this topology on E_{Δ_n} the *adic topology.*

For a lift to $\mathcal{A}_{\Delta_n}$, consider the topology for which the sets

$$(z + U_{\ell,m})_{\ell \in \mathbb{Z}, m \in \mathbb{N}}$$

form a fundamental system of open neighbourhoods of $z \in \mathcal{A}_{\Delta_n}$, where

$$U_{\ell,m} \coloneqq X_{\Delta_n}^{\ell} \mathcal{O}_{\Delta_n}^{+} + \pi^m \mathcal{A}_{\Delta_n}.$$

We call this topology on $\mathcal{A}_{\Delta_n}$ the adic topology as well. The theory of Section 1.4 applies after we prove the following lemma.

Lemma 2.13. *The ring $\mathcal{A}_{\Delta_n}$ is a topological ring for the adic topology.*

Proof. Let $y, z \in \mathcal{A}_{\Delta_n}$. For the open neighbourhood $y + z + U_{\ell,m}$ of $y + z$ note that

$$y + U_{\ell,m} + z + U_{\ell,m} \subseteq y + z + U_{\ell,m}$$

since $U_{\ell,m}$ is closed under addition, therefore addition is continuous on $\mathcal{A}_{\Delta_n}$. For multiplication, consider an open neighbourhood $yz + U_{\ell,m}$ of yz for some $\ell, m \in \mathbb{N}$. Using the Laurent series description of $\mathcal{A}_{\Delta_n}$, we can write $y = y_1 + \pi^m y_2$ and $z = z_1 + \pi^m z_2$ where $y_1, z_1 \in X_{\Delta_n}^{-\ell_1} \mathcal{O}_{\Delta_n}^{+}$ for some $\ell_1 \in \mathbb{N}$ and $y_2, z_2 \in \mathcal{A}_{\Delta_n}$. We then have that

$$\begin{aligned} yU_{\ell+\ell_1,m} &= (y_1 + \pi^m y_2)\left(X_{\Delta_n}^{\ell_1+\ell} \mathcal{O}_{\Delta_n}^{+} + \pi^m \mathcal{A}_{\Delta_n}\right) \\ &\subseteq y_1 X_{\Delta_n}^{\ell_1+\ell} \mathcal{O}_{\Delta_n}^{+} + \pi^m \mathcal{A}_{\Delta_n} \\ &\subseteq X_{\Delta_n}^{-\ell_1+\ell_1+\ell} \mathcal{O}_{\Delta_n}^{+} + \pi^m \mathcal{A}_{\Delta_n} \\ &= U_{\ell,m} \end{aligned}$$

and similarly $zU_{\ell+\ell_1,m} \subseteq U_{\ell,m}$. Furthermore, we also have the inclusion

$$U_{\ell+\ell_1,m} U_{\ell+\ell_1,m} \subseteq U_{\ell,m}$$

since $\ell, \ell_1, m \geq 0$. Therefore

$$(y + U_{\ell+\ell_1,m})(z + U_{\ell+\ell_1,m}) \subseteq yz + U_{\ell,m} + U_{\ell,m} + U_{\ell,m} = yz + U_{\ell,m},$$

meaning that multiplication is continuous on $\mathcal{A}_{\Delta_n}$, as desired. □

Definition 2.14. *For a finitely generated $\mathcal{A}_{\Delta_n}$-module, we call its linear topology the* adic topology *when $\mathcal{A}_{\Delta_n}$ is regarded as a topological ring equipped with the adic topology.*

Our second generalization of the X-adic topology in the multivariable case is the topology on E_{Δ_n} for which the sets $y + E_{\Delta_n,\ell}$ form a fundamental system of open neighbourhoods of $y \in E_{\Delta_n}$ where

$$E_{\Delta_n,\ell} \coloneqq \sum_{\substack{(i_1,\dots,i_n) \in \mathbb{Z}^n \\ i_1+\dots+i_n=\ell}} X_1^{i_1} \dots X_n^{i_n} E_{\Delta_n}^{+}$$

for $\ell \in \mathbb{Z}$. We call this topology on E_{Δ_n} the *weak topology.*

For a lift to $\mathcal{A}_{\Delta_n}$, consider the topology for which the sets

$$(z + \mathcal{V}_{\ell,m})_{\ell \in \mathbb{Z}, m \in \mathbb{N}}$$

form a fundamental system of open neighbourhoods of $z \in \mathcal{A}_{\Delta_n}$, where

$$\mathcal{V}_{\ell,m} := (\mathcal{O}_{\Delta_n})_\ell + \pi^m \mathcal{A}_{\Delta_n}$$

and

$$(\mathcal{O}_{\Delta_n})_\ell := \sum_{\substack{(i_1,\ldots,i_n) \in \mathbb{Z}^n \\ i_1+\ldots+i_n=\ell}} X_1^{i_1} \ldots X_n^{i_n} \mathcal{O}_{\Delta_n}^+.$$

We call this topology on $\mathcal{A}_{\Delta_n}$ the weak topology as well. The theory of Section 1.4 applies for this topology as well after we prove the following lemma.

Lemma 2.15. *The ring $\mathcal{A}_{\Delta_n}$ is a topological ring for the weak topology.*

Proof. Let $y, z \in \mathcal{A}_{\Delta_n}$. For the open neighbourhood $y + z + \mathcal{V}_{\ell,m}$ of $y + z$ note that

$$y + \mathcal{V}_{\ell,m} + z + \mathcal{V}_{\ell,m} \subseteq y + z + \mathcal{V}_{\ell,m}$$

since $\mathcal{V}_{\ell,m}$ is closed under addition, therefore addition is continuous on $\mathcal{A}_{\Delta_n}$. For multiplication, consider an open neighbourhood $yz + \mathcal{V}_{\ell,m}$ of yz for some $\ell, m \in \mathbb{N}$. Using our the Laurent series description of $\mathcal{A}_{\Delta_n}$, we can write $y = y_1 + \pi^m y_2$ and $z = z_1 + \pi^m z_2$ where $y_1, z_1 \in (\mathcal{O}_{\Delta_n})_{-\ell_1}$ for some $\ell_1 \in \mathbb{N}$ and $y_2, z_2 \in \mathcal{A}_{\Delta_n}$. We then have that

$$\begin{aligned}
y\mathcal{V}_{\ell+\ell_1,m} &= (y_1 + \pi^m y_2)\left((\mathcal{O}_{\Delta_n})_{\ell+\ell_1} + \pi^m \mathcal{A}_{\Delta_n}\right) \\
&\subseteq y_1(\mathcal{O}_{\Delta_n})_{\ell+\ell_1} + \pi^m \mathcal{A}_{\Delta_n} \\
&\subseteq (\mathcal{O}_{\Delta_n})_{\ell+\ell_1-\ell_1} + \pi^m \mathcal{A}_{\Delta_n} \\
&= \mathcal{V}_{\ell,m}
\end{aligned}$$

and similarly $z\mathcal{V}_{\ell+\ell_1,m} \subseteq \mathcal{V}_{\ell,m}$. Furthermore, we also have the inclusion

$$\mathcal{V}_{\ell+\ell_1,m}\mathcal{V}_{\ell+\ell_1,m} \subseteq \mathcal{V}_{\ell,m}$$

since $\ell, \ell_1, m \geq 0$. Therefore

$$(y + \mathcal{V}_{\ell+\ell_1,m})(z + \mathcal{V}_{\ell+\ell_1,m}) \subseteq yz + \mathcal{V}_{\ell,m} + \mathcal{V}_{\ell,m} + \mathcal{V}_{\ell,m} = yz + \mathcal{V}_{\ell,m},$$

meaning that multiplication is continuous on $\mathcal{A}_{\Delta_n}$, as desired. □

Definition 2.16. *For a finitely generated $\mathcal{A}_{\Delta_n}$-module, we call its linear topology the* weak topology *when $\mathcal{A}_{\Delta_n}$ is regarded as a topological ring equipped with the weak topology.*

2.4 The categories and statement of the goal

Given a finitely generated $\mathcal{O}_L$-module T, one can equip it with the π-adic topology, that is the topology for which the sets $(\pi^m T)_{m\in\mathbb{N}_{>0}}$ form a fundamental system of open neighbourhoods of zero. This topology on T coincides with the linear topology described in Section 1.4 for $R = \mathcal{O}_L$ with the π-adic topology.

We can now define the first category of interest to us.

Definition 2.17. *A continuous $\mathcal{O}_L$-representation of $G_{\Delta_n,L}$ is a finitely generated $\mathcal{O}_L$-module T equipped with an $\mathcal{O}_L$-linear action*

$$G_{\Delta_n,L} \times T \longrightarrow T$$

which is continuous with respect to the π-adic topology on T. A morphism between two such representations is a $G_{\Delta_n,L}$-equivariant $\mathcal{O}_L$-linear map.

Let $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ denote the category of continuous $\mathcal{O}_L$-representations of $G_{\Delta_n,L}$. For $n = 1$, we obtain the category $\mathrm{Rep}_{\mathcal{O}_L}(G_L)$ of continuous representations of G_L with coefficients in $\mathcal{O}_L$.

Lemma 2.18. *The category $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ is abelian and the forgetful functor into the category of $\mathcal{O}_L$-modules is exact.*

Proof. Let T_1 and T_2 be two objects in $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. Consider the direct sum $T_1 \oplus T_2$ equipped with the diagonal action of $G_{\Delta_n,L}$. Because T_1 and T_2 are continuous, the map

$$\begin{aligned} G_{\Delta_n,L} \times (T_1 \oplus T_2) &\to T_1 \oplus T_2 \\ (\sigma, (v_1, v_2)) &\mapsto (\sigma(v_1), \sigma(v_2)) \end{aligned}$$

is continuous when $T_1 \oplus T_2$ is equipped with the product of the π-adic topologies of T_1 and T_2. By Lemma 1.44 this topology coincides with the π-adic topology of $T_1 \oplus T_2$. Clearly $T_1 \oplus T_2$ is finitely generated over $\mathcal{O}_L$ and therefore $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ has binary products and coproducts which coincide with each other.

Let $f \in \mathrm{Mor}_{\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})}(T_1, T_2)$. We are left to show that $\ker(f)$ and $\mathrm{coker}(f)$ formed in the category of $\mathcal{O}_L$-modules are in fact objects of $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. The equivariance of f implies that both admit an $\mathcal{O}_L$-linear action of $G_{\Delta_n,L}$.

As T_1 is finitely generated over $\mathcal{O}_L$, note that $\ker(f)$ must also be finitely generated over $\mathcal{O}_L$ by Noetherianity. The action of $G_{\Delta_n,L}$ on $\ker(f)$ is continuous for the subspace topology of the π-adic topology on T_1. This subspace topology coincides with the π-adic topology on $\ker(f)$ by the Artin-Rees lemma.

Note that $\mathrm{coker}(f)$ is finitely generated over $\mathcal{O}_L$ because T_2 is so. The action of $G_{\Delta_n,L}$ on $\mathrm{coker}(f)$ is continuous for the quotient topology of the π-adic topology of T_2. This topology coincides with the π-adic topology on $\operatorname{coker}(f)$ by Lemma 1.46. $\square$

Remark 2.19. Let T be an object in $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. For $m \in \mathbb{N}_{>0}$, multiplication by π^m is an endomorphism of T in $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, therefore by Lemma 2.18 we also have that $\pi^m T, T/\pi^m T$ and $T[\pi^m]$ are in $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ as well, where the latter denotes the set of elements of T annihilated by π^m. Because T is finitely generated over $\mathcal{O}_L$, we know that the set of torsion elements of T denoted T^{tor}, equals $T[\pi^m]$ for a large enough $m \in \mathbb{N}_{>0}$. Therefore T^{tor} is in $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ as well.

When we try to prove a statement regarding a representation of $G_{\Delta_n,L}$ with coefficients in $\mathcal{O}_L$, a common strategy is to reduce the statement to the case when the representation is annihilated by a power of π. An important fact that helps us achieve this is the following.

Proposition 2.20. *Let T be an object in $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. Then $\varprojlim_{m\geq 1} T/\pi^m T$ exists in $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ and the natural map*

$$T \to \varprojlim_{m\geq 1} T/\pi^m T$$

is an isomorphism in $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$.

Proof. Let $S \in \operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ together with $G_{\Delta_n,L}$-equivariant $\mathcal{O}_L$-linear maps

$$(f_m : S \to T/\pi^m T)_{m\geq 1}$$

that are compatible with the transition maps. In the category of $\mathcal{O}_L$-modules $\varprojlim_{m\geq 1} T/\pi^m T$ exists and the result of [Sta18, Tag 0316] shows that the natural map from T into $\varprojlim_{m\geq 1} T/\pi^m T$ is an isomorphism of $\mathcal{O}_L$-modules since $\mathcal{O}_L$ is a π-adically complete Noetherian ring and T is finitely generated over $\mathcal{O}_L$. Alternatively, this can also be seen from the classification of finitely generated modules over a DVR and the completeness of $\mathcal{O}_L$.

Thus, there exists a unique $\mathcal{O}_L$-linear map $f : S \to T$ making all the relevant diagrams commute. Let $y \in S, \sigma \in G_{\Delta_n,L}$ and $\operatorname{pr}_m : T \to T/\pi^m T$ be the projection map for $m \in \mathbb{N}_{\geq 1}$. Then for every $m \in \mathbb{N}_{\geq 1}$ we have that

$$\operatorname{pr}_m(f(\sigma(y))) = f_m(\sigma(y)) = \sigma(f_m(y)) = \sigma(\operatorname{pr}_m(f(y))) = \mathrm{pr}_{\mathrm{m}}(\sigma(f(y)))$$

where the second equality is due to the $G_{\Delta_n,L}$-equivariance of f_m. Using the π-adic completeness of T in the category of $\mathcal{O}_L$-modules, it follows that $f(\sigma(y)) = \sigma(f(y))$, hence f is $G_{\Delta_n,L}$-equivariant and T satisfies the universal property of $\varprojlim_{m\geq 1} T/\pi^m T$ in $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. $\square$

We now introduce the second category of interest to us.

Definition 2.21. *A $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-module over $\mathcal{A}_{\Delta_n}$ is a finitely generated module D over $\mathcal{A}_{\Delta_n}$ equipped with a semilinear action of $\mathcal{T}_{+,\Delta_n,L}$. It is called* étale *if for each $i \in \Delta_n$, the linearized map*

$$\begin{aligned} \varphi_i^{\mathrm{lin}} : \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D &\longrightarrow D \\ a \otimes d &\longmapsto a\varphi_i(d) \end{aligned}$$

is an isomorphism.

Definition 2.22. *A morphism between two $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over $\mathcal{A}_{\Delta_n}$ is an $\mathcal{A}_{\Delta_n}$-linear map which commutes with the action of $\mathcal{T}_{+,\Delta_n,L}$ on both modules.*

Let $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ denote the category of étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over $\mathcal{A}_{\Delta_n}$. For $n = 1$, up to a continuity condition we obtain the category $\mathrm{Mod}^{\text{ét}}(\varphi_L, \Gamma_L, \mathcal{A}_L)$ of étale (φ_L, Γ_L)-modules over $\mathcal{A}_L$ from [Sch17]. We explore the continuity condition in more detail in Section 2.5.

Example 2.23. The ring $\mathcal{A}_{\Delta_n}$ considered as a module over itself with the action of $\mathcal{T}_{+,\Delta_n,L}$ defined in Section 2.1 is in $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. Indeed, for $i \in \Delta_n$ the maps

$$\begin{aligned} \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} \mathcal{A}_{\Delta_n} &\longrightarrow \mathcal{A}_{\Delta_n} \\ a \otimes b &\longmapsto a\varphi_i(b) \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}_{\Delta_n} &\longrightarrow \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} \mathcal{A}_{\Delta_n} \\ a &\longmapsto a \otimes 1 \end{aligned}$$

are well defined $\mathcal{A}_{\Delta_n}$-linear maps which revert each other.

Lemma 2.24. *The category $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ is abelian and the forgetful functor into the category of $\mathcal{A}_{\Delta_n}$-modules is exact.*

Proof. Suppose that D_1 and D_2 are objects in $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. Then $D_1 \oplus D_2$ has the structure of an étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-module over $\mathcal{A}_{\Delta_n}$ if we define

$$\tau(d_1, d_2) \coloneqq (\tau(d_1), \tau(d_2))$$

for every $d_1 \in D_1$, $d_2 \in D_2$, $\tau \in \mathcal{T}_{+,\Delta_n,L}$. Indeed, the module $D_1 \oplus D_2$ is finitely generated over $\mathcal{A}_{\Delta_n}$ and the action of every $\tau \in \mathcal{T}_{+,\Delta_n,L}$ is well defined and semilinear. We are left to show that the map

$$\begin{aligned} \varphi_i^{\mathrm{lin}} : \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} (D_1 \oplus D_2) &\longrightarrow D_1 \oplus D_2 \\ a \otimes (d_1, d_2) &\longmapsto a(\varphi_i(d_1), \varphi_i(d_2)) \end{aligned}$$

is bijective for all $i \in \Delta_n$. For this we consider the maps

$$f : \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} (D_1 \oplus D_2) \longrightarrow (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_1) \oplus (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_2)$$

$$a \otimes (d_1, d_2) \longmapsto (a \otimes d_1, a \otimes d_2)$$

and

$$\varphi_{i,D_1}^{\text{lin}} \oplus \varphi_{i,D_2}^{\text{lin}} : (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_1) \oplus (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_2) \longrightarrow D_1 \oplus D_2$$

$$(a_1 \otimes d_1, a_2 \otimes d_2) \longmapsto a_1 \varphi_i(d_1) \otimes a_2 \varphi_i(d_2).$$

Then $\varphi_i^{\text{lin}} = (\varphi_{i,D_1}^{\text{lin}} \oplus \varphi_{i,D_2}^{\text{lin}}) \circ f$. The map f is well defined, because

$$\begin{aligned} f(\varphi_i(b)a \otimes (d_1, d_2)) &= (\varphi_i(b)a \otimes d_1, \varphi_i(b)a \otimes d_2) \\ &= (a \otimes bd_1, a \otimes bd_2) \\ &= f(a \otimes b(d_1, d_2)) \end{aligned}$$

holds for every $a, b \in \mathcal{A}_{\Delta_n}, d_1 \in D_1$ and $d_2 \in D_2$. The map

$$\tilde{f} : (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_1) \oplus (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_2) \longrightarrow \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} (D_1 \oplus D_2)$$

$$(a_1 \otimes d_1, a_2 \otimes d_2) \mapsto a_1 \otimes (d_1, 0) + a_2 \otimes (0, d_2)$$

reverts f and is well defined because

$$\begin{aligned} \tilde{f}(\varphi_i(b_1)a_1 \otimes d_1, \varphi_i(b_2)a_2 \otimes d_2) &= \varphi_i(b_1)a_1 \otimes (d_1, 0) + \varphi_i(b_2)a_2 \otimes (0, d_2) \\ &= a_1 \otimes (b_1 d_1, 0) + a_2 \otimes (0, b_2 d_2) \\ &= \tilde{f}(a_1 \otimes b_1 d_1, a_2 \otimes b_2 d_2) \end{aligned}$$

holds for all $a_1, a_2, b_1, b_2 \in \mathcal{A}_{\Delta_n}, d_1 \in D_1$ and $d_2 \in D_2$. Therefore f is a bijection and by the étale property of D_1 and D_2, $\varphi_{i,D_1}^{\text{lin}} \oplus \varphi_{i,D_2}^{\text{lin}}$ is a bijection as well. As φ_i^{lin} is a composition of two bijections, it must be a bijection as well. Therefore $\text{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ has binary products and coproducts which coincide with each other.

Let $g : D_1 \to D_2$ be a morphism of étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over $\mathcal{A}_{\Delta_n}$. We are left to show that $\ker(g)$ and $\text{coker}(g)$ formed in the category of $\mathcal{A}_{\Delta_n}$-modules are in fact objects of $\text{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. The Noetherianity of $\mathcal{A}_{\Delta_n}$ implies that $\ker(g)$ is finitely generated over $\mathcal{A}_{\Delta_n}$ and $\text{coker}(g)$ is also finitely generated over $\mathcal{A}_{\Delta_n}$, being a quotient of D_2. Also, clearly both $\ker(g)$ and $\text{coker}(g)$ admit a semilinear action of $\mathcal{T}_{+,\Delta_n,L}$. To see that they are étale, for $i \in \Delta_n$ one considers the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} \ker(g) \arrow[r, "\varphi^{\mathrm{lin}}_{i,\ker(g)}"] \arrow[d] & \ker(g) \arrow[d] \\
\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_1 \arrow[r, "\varphi^{\mathrm{lin}}_{i,D_1}"] \arrow[d] & D_1 \arrow[d] \\
\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_2 \arrow[r, "\varphi^{\mathrm{lin}}_{i,D_2}"] \arrow[d] & D_2 \arrow[d] \\
\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} \mathrm{coker}(g) \arrow[r, "\varphi^{\mathrm{lin}}_{i,\mathrm{coker}(g)}"] \arrow[d] & \mathrm{coker}(g) \arrow[d] \\
0 & 0
\end{tikzcd}$$

whose left column is exact by Corollary 2.3. The two middle horizontal arrows are isomorphisms by the étale property of D_1 and D_2, therefore the top and bottom horizontal arrows also have to be isomorphisms and the conclusion follows. $\square$

Remark 2.25. Let D be an object in $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. For $m \in \mathbb{N}_{>0}$, multiplication by π^m is an endomorphism of D in $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$, therefore by Lemma 2.24 we also have that $\pi^m D$, $D/\pi^m D$ and $D[\pi^m]$ are in $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ as well.

We have the following counterpart of Proposition 2.20 in the category $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$.

Proposition 2.26. *Let D be an object in $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. Then $\varprojlim_{m\geq 1} D/\pi^m D$ exists in $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ and the natural map*

$$D \to \varprojlim_{m\geq 1} D/\pi^m D$$

is an isomorphism in $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$.

Proof. In the category of $\mathcal{A}_{\Delta_n}$-modules $\varprojlim_{m\geq 1} D/\pi^m D$ exists and the result of [Sta18, Tag 00MA] shows that the natural map from D into $\varprojlim_{m\geq 1} D/\pi^m D$ is an isomorphism of $\mathcal{A}_{\Delta_n}$-modules since $\mathcal{A}_{\Delta_n}$ is a π-adically complete Noetherian ring by Lemma 2.1 and D is finitely generated over $\mathcal{A}_{\Delta_n}$. The rest of the proof is analogous to that of Proposition 2.20. $\square$

Lemma 2.27. *Suppose that $D_1, D_2 \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. Then $D_1 \otimes_{\mathcal{A}_{\Delta_n}} D_2$ can be regarded as an object of $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ by defining*

$$\tau(d_1 \otimes d_2) := \tau(d_1) \otimes \tau(d_2)$$

on pure tensors for all $d_1 \in D_1$, $d_2 \in D_2$ and $\tau \in \mathcal{T}_{+,\Delta_n,L}$.

Proof. The module $D_1 \otimes_{\mathcal{A}_{\Delta_n}} D_2$ is finitely generated over $\mathcal{A}_{\Delta_n}$ and the action of every $\tau \in \mathcal{T}_{+,\Delta_n,L}$ is well defined and semilinear. We are left to show that the map

$$\begin{aligned} \varphi_i^{\mathrm{lin}} : \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} (D_1 \otimes_{\mathcal{A}_{\Delta_n}} D_2) &\longrightarrow D_1 \otimes_{\mathcal{A}_{\Delta_n}} D_2 \\ a \otimes (d_1 \otimes d_2) &\longmapsto a\big(\varphi_i(d_1) \otimes \varphi_i(d_2)\big) \end{aligned}$$

is bijective for all $i \in \Delta_n$. We factor φ_i^{lin} through the map

$$f : \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} (D_1 \otimes_{\mathcal{A}_{\Delta_n}} D_2) \longrightarrow (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_1) \otimes_{\mathcal{A}_{\Delta_n}} (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_2)$$

$$a \otimes (d_1 \otimes d_2) \longmapsto a\big((1 \otimes d_1) \otimes (1 \otimes d_2)\big)$$

and the map

$$\varphi_{i,D_1}^{\mathrm{lin}} \otimes \varphi_{i,D_2}^{\mathrm{lin}} : (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_1) \otimes_{\mathcal{A}_{\Delta_n}} (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_2) \longrightarrow D_1 \otimes_{\mathcal{A}_{\Delta_n}} D_2$$

$$(a_1 \otimes d_1) \otimes (a_2 \otimes d_2) \longmapsto a_1\varphi_i(d_1) \otimes a_2\varphi_i(d_2).$$

Then $\varphi_i^{\mathrm{lin}} = (\varphi_{i,D_1}^{\mathrm{lin}} \otimes \varphi_{i,D_2}^{\mathrm{lin}}) \circ f$. The map f is well defined, because

$$\begin{aligned} f(\varphi_i(c)a \otimes (d_1 \otimes d_2)) &= \varphi_i(c)a\big((1 \otimes d_1) \otimes (1 \otimes d_2)\big) \\ &= (\varphi_i(c)a \otimes d_1) \otimes (1 \otimes d_2) \\ &= (a \otimes cd_1) \otimes (1 \otimes d_2) \\ &= f(a \otimes (cd_1 \otimes d_2)) \end{aligned}$$

for every $d_1 \in D_1, d_2 \in D_2$ and $a, c \in \mathcal{A}_{\Delta_n}$. The inverse of f is the map

$$g : (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_1) \otimes_{\mathcal{A}_{\Delta_n}} (\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D_2) \longrightarrow \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} (D_1 \otimes_{\mathcal{A}_{\Delta_n}} D_2)$$

$$(a_1 \otimes d_1) \otimes (a_2 \otimes d_2) \longmapsto a_1a_2 \otimes (d_1 \otimes d_2)$$

which is well defined, because for every $c_1, c_2 \in \mathcal{A}_{\Delta_n}$ we have the equalities

$$\begin{aligned} g((\varphi_i(c_1)a_1 \otimes d_1) \otimes (\varphi_i(c_2)a_2 \otimes d_2)) &= \varphi_i(c_1)\varphi_i(c_2)a_1a_2 \otimes (d_1 \otimes d_2) \\ &= a_1a_2 \otimes (c_1d_1 \otimes c_2d_2) \\ &= g((a_1 \otimes c_1d_1) \otimes (a_2 \otimes c_2d_2)). \end{aligned}$$

Since f is bijective and $\varphi_{i,D_1}^{\mathrm{lin}} \otimes \varphi_{i,D_2}^{\mathrm{lin}}$ is bijective because D_1 and D_2 are étale, the conclusion follows. □

Example 2.28. Suppose that $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ is a free module over $\mathcal{A}_{\Delta_n}$ of rank m. Let

$$\det(D) \coloneqq \bigwedge^m D$$

be the top exterior power of D. We claim that

$$\det(D) \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$$

with the action of $\mathcal{T}_{+,\Delta_n,L}$ explained below. By Lemma 2.27 $D^{\otimes m}$ is an étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-module if for $\tau \in \mathcal{T}_{+,\Delta_n,L}$ and $d_1, \ldots, d_m \in D$ we define

$$\tau(d_1 \otimes \ldots \otimes d_m) \coloneqq \tau(d_1) \otimes \ldots \otimes \tau(d_m).$$

Let $W \subseteq D^{\otimes m}$ be the $\mathcal{A}_{\Delta_n}$-submodule generated by all the pure tensors $d_1 \otimes \ldots \otimes d_m$ such that $d_i = d_j$ for some $i \neq j$. Then

$$\det(D) = D^{\otimes m}/W.$$

If $d_i = d_j$ for some $i \neq j$, it follows that $\tau(d_i) = \tau(d_j)$ so τ preserves W. Hence there is a well defined semilinear action of $\mathcal{T}_{+,\Delta_n,L}$ on $\det(D)$. Let $\{x_1, \ldots, x_m\}$ be a basis of D. Then $\{x_1 \wedge \ldots \wedge x_m\}$ is a basis of $\det(D)$. To show that $\det(D)$ is étale, it suffices to check that $\tau(x_1 \wedge \ldots \wedge x_m) = c(x_1 \wedge \ldots \wedge x_m)$ for some $c \in \mathcal{A}_{\Delta_n}^{\times}$. We know that $c = \det(A_\tau)$ where A_τ is the matrix of the linear map

$$\begin{aligned} \tau^{\mathrm{lin}} : \mathcal{A}_{\Delta_n} \otimes_{\tau, \mathcal{A}_{\Delta_n}} D &\longrightarrow D \\ a \otimes d &\longmapsto a\tau(d) \end{aligned}$$

with respect to the bases $\{1 \otimes x_1, \ldots, 1 \otimes x_m\}$ and $\{x_1, \ldots, x_m\}$, respectively. Since D is étale, it follows that $\det(A_\tau) \in \mathcal{A}_{\Delta_n}^{\times}$ and we are done.

Remark 2.29. In the following, we will also study in detail the étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules D over $\mathcal{A}_{\Delta_n}$ for which $\pi D = 0$. Clearly D is an E_{Δ_n}-module and we have an isomorphism $\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} D \simeq E_{\Delta_n} \otimes_{\varphi_i, E_{\Delta_n}} D$ for every $i \in \Delta_n$. The étale condition on D means that we require the map

$$\begin{aligned} \varphi_i^{\mathrm{lin}} : E_{\Delta_n} \otimes_{\varphi_i, E_{\Delta_n}} D &\to D \\ e \otimes d &\mapsto e\varphi_i(d) \end{aligned}$$

to be an isomorphism for every $i \in \Delta_n$, where $e \in E_{\Delta_n}$ and $d \in D$. The objects in $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ annihilated by π and their morphisms form a subcategory which we denote $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$.

Remark 2.30. Lemma 2.11 and Lemma 2.12 imply that every object in $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ is a stably free E_{Δ_n}-module.

The main goal of the thesis is to show that there is an equivalence of categories between $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$.

2.5 The continuity condition

In the definition of (φ, Γ)-modules it is customary in the literature, like [FO10] or [Sch17], to require that the group action is *continuous* for a certain topology on the underlying objects. For example, compared to Definition 2.21, in [Sch17], for a

(φ_L, Γ_L)-module over $\mathcal{A}_L$ it is required in addition that the underlying action of Γ_L is continuous for the weak topology of the module. The latter refers to the linear topology of the module for the ring $\mathcal{A}_L$ equipped with the topology described at the beginning of Section 2.3.

Knowing that for every $M \in \mathrm{Mod}^{\text{ét}}(\varphi_L, \Gamma_L, \mathcal{A}_L)$ the map

$$\Gamma_L \times M \longrightarrow M$$

is continuous for the weak topology on M, played an important role in [Sch17] in proving that the functor from $\mathrm{Mod}^{\text{ét}}(\varphi_L, \Gamma_L, \mathcal{A}_L)$ into $\mathrm{Rep}_{\mathcal{O}_L}(G_L)$ maps each M into a *continuous* representation of G_L over $\mathcal{O}_L$.

However, in Section 2.2 of [Sch17] it is also independently proven that if M is an *étale* (φ_L, Γ_L)-module over $\mathcal{A}_L$, then the underlying group action is automatically continuous for the weak topology on M. The purpose of this section is to show more generally, that if $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$, then the map

$$\Gamma_{\Delta_n,L} \times D \longrightarrow D$$

is continuous for the weak topology on D following a similar strategy to that in [Sch17], hence we can avoid including a continuity condition in $\mathrm{Mod}^{\text{ét}}(\varphi_L, \Gamma_L, \mathcal{A}_L)$.

2.5.1 Reduction to the mod π^m case

Like in the proof of the statement in the one variable case in [Sch17], we first prove that it suffices to show that the action of $\Gamma_{\Delta_n,L}$ is continuous on objects of $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ annihilated by a power of π.

Proposition 2.31. *Let $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. The action of $\Gamma_{\Delta_n,L}$ on D is continuous with respect to the weak topology on D if and only if for every $m > 0$ the action of $\Gamma_{\Delta_n,L}$ on $D/\pi^m D$ is continuous with respect to the weak topology on $D/\pi^m D$.*

Proof. Suppose that the map

$$\Gamma_{\Delta_n,L} \times D \to D$$

is continuous and let $m > 0$. Consider the commutative diagram

$$\begin{tikzcd} \Gamma_{\Delta_n,L} \times D \arrow[r, "\psi"] \arrow[d, "\mathrm{id} \times \mathrm{pr}_m"] & D \arrow[d, "\mathrm{pr}_m"] \\ \Gamma_{\Delta_n,L} \times D/\pi^m D \arrow[r, "\psi_m"] & D/\pi^m D \end{tikzcd} \tag{2.8}$$

whose horizontal arrows are the maps induced by the group action and where

$$\mathrm{pr}_m : D \to D/\pi^m D$$

is the natural projection map. Let $U \subseteq D/\pi^m D$ be an open set. The left vertical arrow of (2.8) is a continuous surjective open map by Lemma 1.46, thus a quotient map. Then to show that $\psi_m^{-1}(U) \subseteq \Gamma_{\Delta_n,L} \times D/\pi^m D$ is open, we need to show that

$$(\mathrm{id} \times \mathrm{pr}_m)^{-1}(\psi_m^{-1}(U)) \subseteq \Gamma_{\Delta_n,L} \times D$$

is open. We know that

$$(\mathrm{id} \times \mathrm{pr}_m)^{-1}(\psi_m^{-1}(U)) = \psi^{-1}(\mathrm{pr}_m^{-1}(U))$$

and because ψ and pr_m are continuous, the action of $\Gamma_{\Delta_n,L}$ on $D/\pi^m D$ is continuous.

Now assume that the action of $\Gamma_{\Delta_n,L}$ on $D/\pi^m D$ is continuous for every $m > 0$ and let $d_1, \ldots, d_s$ be a generating set of D. By Lemma 1.43 the sets

$$W_{\ell,m} := \mathcal{V}_{\ell,m} d_1 + \ldots + \mathcal{V}_{\ell,m} d_s$$

form a fundamental system of open neighbourhoods of zero for the weak topology of D where $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$. Let $\sigma \in \Gamma_{\Delta_n,L}$ and $x \in D$. It suffices to prove that

$$\psi^{-1}(\sigma(x) + W_{\ell,m}) \subseteq \Gamma_{\Delta_n,L} \times D$$

is open. It is clear that

$$\begin{aligned}
W_{\ell,m} &= ((\mathcal{O}_{\Delta_n})_\ell + \pi^m \mathcal{A}_{\Delta_n}) d_1 + \ldots + ((\mathcal{O}_{\Delta_n})_\ell + \pi^m \mathcal{A}_{\Delta_n}) d_s \\
&\supseteq \pi^m \mathcal{A}_{\Delta_n} d_1 + \ldots + \pi^m \mathcal{A}_{\Delta_n} d_s \\
&= \pi^m D,
\end{aligned}$$

therefore $\sigma(x) + W_{\ell,m} = \mathrm{pr}_m^{-1}(\mathrm{pr}_m(\sigma(x) + W_{\ell,m}))$ and by (2.8) we have that

$$\begin{aligned}
\psi^{-1}(\sigma(x) + W_{\ell,m}) &= \psi^{-1}(\mathrm{pr}_m^{-1}(\mathrm{pr}_m(\sigma(x) + W_{\ell,m}))) \\
&= (\mathrm{id} \times \mathrm{pr}_m)^{-1}(\psi_m^{-1}(\mathrm{pr}_m(\sigma(x) + W_{\ell,m}))).
\end{aligned}$$

The latter is open since $\mathrm{pr}_m(\sigma(x) + W_{\ell,m}) \subseteq D/\pi^m D$ is open by Lemma 1.46, ψ_m is continuous by assumption and $\mathrm{id} \times \mathrm{pr}_m$ is continuous. □

2.5.2 The Colmez module D^{++}

For $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ annihilated by a power of π we will show that there exists a fundamental system of open neighbourhoods of zero for the weak topology in which every set is preserved by the action of $\Gamma_{\Delta_n,L}$ and whose union covers D.

To define these neighbourhoods one introduces the analog of the Colmez module D^{++} from Remark II.1 of [Col10] which is defined using the *adic* topology in the following way.

Definition 2.32. *For $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ let*

$$D^{++} := \{x \in D : \lim_{k\to\infty} \varphi^k(x) = 0 \textit{ for the adic topology on } D\}$$

where by $\lim_{k\to\infty} \varphi^k(x) = 0$ we mean that for every open neighbourhood U of zero in D with respect to the adic topology, there exists a large enough $k_0 \in \mathbb{N}$ such that $\varphi^k(x) \in U$ for all $k \geq k_0$.

Lemma 2.33. *Let $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. Then we have the following:*

(i) D^{++} is an $\mathcal{O}^+_{\Delta_n}$-submodule of D.

(ii) For every $\tau \in \mathcal{T}_{+,\Delta_n,L}$ we have that $\tau(D^{++}) \subseteq D^{++}$.

(iii) We have that

$$\sigma\left(X_1^{i_1} \dots X_n^{i_n} D^{++}\right) \subseteq X_1^{i_1} \dots X_n^{i_n} D^{++}$$

for every $i_1, \dots, i_n \in \mathbb{Z}$ and $\sigma \in \Gamma_{\Delta_n,L}$.

Proof. (i) Let $d_1, \dots, d_r$ be a generating set of D over $\mathcal{A}_{\Delta_n}$. By Lemma 1.43 the sets

$$\mathcal{W}_{\ell,m} := U_{\ell,m} d_1 + \dots + U_{\ell,m} d_r$$

form a fundamental system of open neighbourhoods of zero in D for the adic topology, where $\ell \in \mathbb{Z}, m \in \mathbb{N}$. Hence if $x, y \in D^{++}$, given any $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$, there exists a large enough $k_0 \in \mathbb{N}$ such that $\varphi^k(x), \varphi^k(y) \in \mathcal{W}_{\ell,m}$ for every $k \geq k_0$. Since $\mathcal{W}_{\ell,m}$ is an $\mathcal{O}^+_{\Delta_n}$-module, we have that

$$\varphi^k(x+y) = \varphi^k(x) + \varphi^k(y) \in \mathcal{W}_{\ell,m},$$

as well as

$$\varphi^k(ax) = \varphi^k(a)\varphi^k(x) \in \mathcal{W}_{\ell,m}$$

for $a \in \mathcal{O}^+_{\Delta_n}$, because $\varphi^k(a) \in \mathcal{O}^+_{\Delta_n}$. Therefore D^{++} is an $\mathcal{O}^+_{\Delta_n}$-submodule of D.

(ii) We first show that $\tau : D \longrightarrow D$ is a continuous map with respect to the adic topology. For our generating set of D from part (i), for every $1 \leq j \leq r$ we write

$$\tau(d_j) = \sum_{k=1}^{r} a_{jk} d_k$$

for some $a_{jk} \in \mathcal{A}_{\Delta_n}$. Given $\ell, m \in \mathbb{N}$, we write

$$a_{jk} = b_{jk} + \pi^m c_{jk}$$

for some $b_{jk} \in \mathcal{O}_{\Delta_n}$ and $c_{jk} \in \mathcal{A}_{\Delta_n}$. Let $s \in \mathbb{N}$ be large enough such that $b_{jk} \in X_{\Delta_n}^{-s}\mathcal{O}_{\Delta_n}^{+}$ for every j, k. We also have that $\tau(X_{\Delta_n}\mathcal{O}_{\Delta_n}^{+}) \subseteq X_{\Delta_n}\mathcal{O}_{\Delta_n}^{+}$ and thus

$$\begin{aligned}
\tau\left(\mathcal{W}_{\ell+s,m}\right) &= \tau\left((X_{\Delta_n}^{\ell+s}\mathcal{O}_{\Delta_n}^{+} + \pi^m\mathcal{A}_{\Delta_n})d_1 + \ldots + (X_{\Delta_n}^{\ell+s}\mathcal{O}_{\Delta_n}^{+} + \pi^m\mathcal{A}_{\Delta_n})d_r\right) \\
&= \tau\left(X_{\Delta_n}^{\ell+s}\mathcal{O}_{\Delta_n}^{+}d_1 + \ldots + X_{\Delta_n}^{\ell+s}\mathcal{O}_{\Delta_n}^{+}d_r\right) + \pi^m\tau(D) \\
&= \tau\left(X_{\Delta_n}\right)^{\ell+s}\tau\left(\mathcal{O}_{\Delta_n}^{+}d_1 + \ldots + \mathcal{O}_{\Delta_n}^{+}d_r\right) + \pi^m\tau(D) \\
&\subseteq X_{\Delta_n}^{\ell+s-s}\left(\mathcal{O}_{\Delta_n}^{+}d_1 + \ldots + \mathcal{O}_{\Delta_n}^{+}d_r\right) + \pi^m D \\
&= \mathcal{W}_{\ell,m},
\end{aligned}$$

implying the continuity of τ. Let now $d \in D^{++}$. Since τ is continuous and commutes with φ,

$$\lim_{k\to\infty}\varphi^k(\tau(d)) = \lim_{k\to\infty}\tau\left(\varphi^k(d)\right) = \tau\left(\lim_{k\to\infty}\left(\varphi^k(d)\right)\right) = \tau(0) = 0$$

thus $\tau(d) \in D^{++}$, as desired.

(iii) For every $\sigma_i \in \Gamma_{i,L}$ and $j \in \mathbb{Z}$, we have that

$$\sigma_i(X_i^j\mathcal{O}_L[\![X_i]\!]) \subseteq X_i^j\mathcal{O}_L[\![X_i]\!],$$

therefore

$$\sigma(X_1^{i_1}\ldots X_n^{i_n}\mathcal{O}_{\Delta_n}^{+}) \subseteq X_1^{i_1}\ldots X_n^{i_n}\mathcal{O}_{\Delta_n}^{+}$$

for every $\sigma \in \Gamma_{\Delta_n,L}$. Combining this with the results of the previous parts, the conclusion follows. $\square$

Lemma 2.34. *For $D_1, D_2 \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ we have*

$$(D_1 \oplus D_2)^{++} = D_1^{++} \oplus D_2^{++}.$$

Proof. The product of the adic topologies of D_1 and D_2 is the adic topology of $D_1 \oplus D_2$ by Lemma 1.44. Therefore the sets $U \times V \subseteq D_1 \oplus D_2$ form a basis of open neighbourhoods of $D_1 \oplus D_2$ if we vary U among the open sets of D_1 and V among the open sets of D_2. For $x_1 \in D_1$ and $x_2 \in D_2$, we have that $\varphi^k(x_1, x_2) \in U \times V$ if and only if $\varphi^k(x_1) \in U$ and $\varphi^k(x_2) \in V$, hence the conclusion follows. $\square$

We provide a sufficient condition for an $\mathcal{O}_{\Delta_n}^{+}$-submodule of D to be contained in D^{++}.

Lemma 2.35. *If M is a finitely generated $\mathcal{O}_{\Delta_n}^{+}$-submodule of D such that*

$$\varphi(M) \subseteq X_{\Delta_n}M,$$

then $M \subseteq D^{++}$.

Proof. From the assumption that $\varphi(M) \subseteq X_{\Delta_n}M$ we deduce that

$$\begin{aligned}
\varphi^{j+1}(M) &= \varphi^j\left(\varphi(M)\right) \\
&\subseteq \varphi^j(X_{\Delta_n}M) \\
&= \varphi^j(X_{\Delta_n})\varphi^j(M) \\
&\subseteq \dots \\
&\subseteq \varphi^j(X_{\Delta_n})\varphi^{j-1}(X_{\Delta_n})\dots\varphi(X_{\Delta_n})X_{\Delta_n}M \\
&\subseteq \varphi^j(X_{\Delta_n})M
\end{aligned} \tag{2.9}$$

for any $j \in \mathbb{N}$. Let $m_1, \dots, m_s$ be generators of the $\mathcal{O}^+_{\Delta_n}$-module M and $x \in M$ be an arbitrary element. By (2.9), for every $j \in \mathbb{N}$ we can write

$$\varphi^{j+1}(x) = \sum_{k=1}^{s} \varphi^j(X_{\Delta_n})a_{jk}m_k$$

for some $a_{jk} \in \mathcal{O}^+_{\Delta_n}$. By Lemma 2.4 we know that $\lim_{j\to\infty} \varphi^j(X_{\Delta_n}) = 0$ for the adic topology. Since $a_{jk} \in \mathcal{O}^+_{\Delta_n}$ and the adic topology of $\mathcal{A}_{\Delta_n}$ has a fundamental system of neighbourhoods of zero given by $\mathcal{O}^+_{\Delta_n}$-modules, we obtain that

$$\lim_{j\to\infty} \varphi^j(X_{\Delta_n})a_{jk} = 0$$

for every $1 \leq k \leq s$. Since D is a topological module for the adic topology by Lemma 1.45, it follows that

$$\lim_{j\to\infty} \varphi^j(x) = 0$$

for the adic topology, therefore $x \in D^{++}$ as desired. □

Corollary 2.36. *Let $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ such that $\pi^m D = 0$ for some $m > 0$. Then*

$$D = D^{++}[X_{\Delta_n}^{-1}].$$

Proof. Let $d_1, \dots, d_s$ be a generating set of D. For every $1 \leq j \leq s$, write

$$\varphi(d_j) = \sum_{k=1}^{s} a_{jk}d_k$$

for some $a_{jk} \in \mathcal{A}_{\Delta_n}$. Write

$$a_{jk} = b_{jk} + \pi^m c_{jk}$$

for some $b_{jk} \in \mathcal{O}_{\Delta_n}$ and $c_{jk} \in \mathcal{A}_{\Delta_n}$. By Lemma 2.5 we can choose $r \in \mathbb{N}$ large enough such that

$$b_{jk}\left(\frac{\varphi(X_{\Delta_n})}{X_{\Delta_n}}\right)^{rp^{m-1}} \in X_{\Delta_n}\mathcal{O}^+_{\Delta_n} + \pi^m\mathcal{A}_{\Delta_n} \tag{2.10}$$

for all j, k. Let

$$M := X_{\Delta_n}^{rp^{m-1}}\left(\mathcal{O}^+_{\Delta_n}d_1 + \dots + \mathcal{O}^+_{\Delta_n}d_s\right).$$

For every $1 \leq j \leq s$ we then have that

$$\begin{aligned}
\varphi\left(X_{\Delta_n}^{rp^{m-1}} d_j\right) &= \left(\varphi\left(X_{\Delta_n}\right)\right)^{rp^{m-1}} \varphi\left(d_j\right) \\
&= (\varphi(X_{\Delta_n}))^{rp^{m-1}} \sum_{k=1}^{s} a_{jk} d_k \\
&= (\varphi(X_{\Delta_n}))^{rp^{m-1}} \sum_{k=1}^{s} (b_{jk} + \pi^m c_{jk}) d_k \\
&= (\varphi(X_{\Delta_n}))^{rp^{m-1}} \sum_{k=1}^{s} b_{jk} d_k \\
&= \sum_{k=1}^{s} \left[\left(\frac{\varphi(X_{\Delta_n})}{X_{\Delta_n}} \right)^{rp^{m-1}} b_{jk} \right] X_{\Delta_n}^{rp^{m-1}} d_k \\
&\in X_{\Delta_n} M
\end{aligned}$$

where the fourth equality uses that $\pi^m D = 0$ and the last inclusion uses (2.10) and $\pi^m D = 0$ again. Therefore

$$\varphi(M) \subseteq X_{\Delta_n} M$$

implying that $M \subseteq D^{++}$ by Lemma 2.35. Note that $\mathcal{A}_{\Delta_n} = \mathcal{O}_{\Delta_n} + \pi^m \mathcal{A}_{\Delta_n}$ and $\pi^m D = 0$ imply that

$$\begin{aligned}
D &= \mathcal{A}_{\Delta_n} d_1 + \ldots + \mathcal{A}_{\Delta_n} d_s \\
&= (\mathcal{O}_{\Delta_n} + \pi^m \mathcal{A}_{\Delta_n}) d_1 + \ldots + (\mathcal{O}_{\Delta_n} + \pi^m \mathcal{A}_{\Delta_n}) d_s \\
&= \mathcal{O}_{\Delta_n} d_1 + \ldots + \mathcal{O}_{\Delta_n} d_s \\
&= M[X_{\Delta_n}^{-1}],
\end{aligned}$$

therefore $D = M[X_{\Delta_n}^{-1}] \subseteq D^{++}[X_{\Delta_n}^{-1}] \subseteq D$ which implies the desired equality. $\square$

Remark 2.37. To simplify notation, for $m \in \mathbb{N}_{>0}$ consider the rings

$$\mathcal{O}_{\Delta_n,m}^{+} := (\mathcal{O}_L/\pi^m \mathcal{O}_L)[\![X_1, \ldots, X_n]\!]$$

and

$$\mathcal{O}_{\Delta_n,m} := \mathcal{O}_{\Delta_n,m}^{+}[X_{\Delta_n}^{-1}] \simeq \mathcal{O}_{\Delta_n}/\pi^m \mathcal{O}_{\Delta_n} \simeq \mathcal{A}_{\Delta_n}/\pi^m \mathcal{A}_{\Delta_n}.$$

Lemma 2.13 and a repeated use of Lemma 1.43 for the adic topology show that $\mathcal{O}_{\Delta_n,m}$ is a topological ring for the adic topology and that the sets

$$(X_{\Delta_n}^{\ell} \mathcal{O}_{\Delta_n,m}^{+})_{\ell \in \mathbb{Z}}$$

form a fundamental system of open neighbourhoods of zero. When $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ is annihilated by π^m, the adic topology of D is the quotient topology induced by any $\mathcal{O}_{\Delta_n,m}$-linear surjection $(\mathcal{O}_{\Delta_n,m})^{\oplus k} \to D$. In particular, if $d_1, \ldots, d_k$ generate D as an $\mathcal{O}_{\Delta_n,m}$-module, the sets

$$(X_{\Delta_n}^{\ell} \mathcal{O}_{\Delta_n,m}^{+} d_1 + \ldots + X_{\Delta_n}^{\ell} \mathcal{O}_{\Delta_n,m}^{+} d_k)_{\ell \in \mathbb{Z}}$$

form a fundamental system of open neighbourhoods of zero for the adic topology on D.

Similarly, Lemma 2.15 and a repeated use of Lemma 1.43 for the weak topology show that $\mathcal{O}_{\Delta_n,m}$ is a topological ring for the weak topology and that the sets

$$(\mathcal{O}_{\Delta_n,m})_\ell := \sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{Z}^n \\ i_1+\ldots+i_n=\ell}} X_1^{i_1}\ldots X_n^{i_n}\mathcal{O}_{\Delta_n,m}^+$$

form a fundamental system of open neighbourhoods of zero for the weak topology where $\ell \in \mathbb{Z}$. If $d_1, \ldots, d_k$ generate D over $\mathcal{O}_{\Delta_n,m}$, the sets

$$((\mathcal{O}_{\Delta_n,m})_\ell d_1 + \ldots + (\mathcal{O}_{\Delta_n,m})_\ell d_k)_{\ell\in\mathbb{Z}}$$

form a fundamental system of open neighbourhoods of zero for the weak topology on D.

We now provide a sufficient condition for a submodule of D to contain D^{++} when D is annihilated by π.

Lemma 2.38. *Let $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ such that $\pi D = 0$ and D is free over E_{Δ_n} with basis $d_1, \ldots, d_r$. If for $M := E_{\Delta_n}^+ d_1 + \ldots + E_{\Delta_n}^+ d_r$ we have that*

$$M \subseteq E_{\Delta_n}^+ \varphi(M)$$

then $D^{++} \subseteq M$.

Proof. Iterating $M \subseteq E_{\Delta_n}^+\varphi(M)$ we obtain that $M \subseteq E_{\Delta_n}^+\varphi^j(M)$ for all $j \geq 1$. Hence, for any $j \geq 1$ and $1 \leq \ell \leq r$, we have that

$$d_\ell = \sum_{k=1}^{r} f_{jk\ell}\varphi^j(d_k)$$

for some $f_{jk\ell} \in E_{\Delta_n}^+$. Let $d \in D^{++}$ be an arbitrary element. We now show that $d \in M$. For any $j \geq 0$, we write

$$\varphi^j(d) = \sum_{\ell=1}^{r} g_{j\ell}d_\ell$$

with $g_{j\ell} \in E_{\Delta_n}$. Since $d \in D^{++}$ and $d_1, \ldots, d_r$ is a basis over E_{Δ_n}, it follows that

$$\lim_{j\to\infty} g_{j\ell} = 0 \tag{2.11}$$

for all $1 \leq \ell \leq r$. Computing $\varphi^j(d)$ for $j \geq 1$ in two different ways, we also obtain that

$$\begin{aligned}\sum_{k=1}^{r} \varphi^j(g_{0k})\varphi^j(d_k) = \varphi^j\left(\sum_{k=1}^{r} g_{0k}d_k\right) &= \varphi^j(d) \\ &= \sum_{\ell=1}^{r} g_{j\ell}d_\ell = \sum_{\ell=1}^{r} g_{j\ell}\left(\sum_{k=1}^{r} f_{jk\ell}\varphi^j(d_k)\right) \\ &= \sum_{k=1}^{r}\sum_{\ell=1}^{r} g_{j\ell}f_{jk\ell}\varphi^j(d_k).\end{aligned}$$

The module D is étale, therefore $\varphi^j(d_1), \ldots, \varphi^j(d_r)$ also form an E_{Δ_n}-basis of D and thus

$$\varphi^j(g_{0k}) = \sum_{\ell=1}^{r} g_{j\ell}f_{jk\ell} \tag{2.12}$$

for all $j \geq 1$. Since $f_{jk\ell} \in E_{\Delta_n}^+$, by (2.11) and (2.12) it follows that

$$\lim_{j\to\infty} \varphi^j(g_{0k}) = 0 \tag{2.13}$$

for all $1 \leq k \leq r$. Because $\varphi^j(g_{0k}) = g_{0k}(X_1^{q^j}, \ldots, X_n^{q^j})$, we have that $g_{0k} \in E_{\Delta_n}^+$ for all $1 \leq k \leq r$ by (2.13). Therefore

$$d = \sum_{k=1}^{r} g_{0k}d_k \in M$$

as desired. □

Proposition 2.39. *Let $D \in \mathrm{Mod}^{ét}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ such that $\pi^m D = 0$ for some $m > 0$. Then D^{++} is a finitely generated module over $\mathcal{O}_{\Delta_n}^+$.*

Proof. The short exact sequence

$$0 \longrightarrow \pi D \longrightarrow D \longrightarrow D/\pi D \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow (\pi D)^{++} \longrightarrow D^{++} \longrightarrow (D/\pi D)^{++}.$$

Assume that $(\pi D)^{++}$ and $(D/\pi D)^{++}$ are finitely generated over $\mathcal{O}_{\Delta_n}^+$. By the Noetherianity of $\mathcal{O}_{\Delta_n}^+$, it follows that the image of $D^{++} \to (D/\pi D)^{++}$ is finitely generated over $\mathcal{O}_{\Delta_n}^+$, therefore so must be D^{++}. The module πD is annihilated by π^{m-1} and $D/\pi D$ is annihilated by π. Then our problem is reduced inductively to the case when $\pi D = 0$, which we assume from now on.

Case 1. Suppose that D is a free module over E_{Δ_n}. Let $d_1, \ldots, d_r$ be a basis of D over E_{Δ_n}. Using the étale condition on D we know that $\varphi(d_1), \ldots, \varphi(d_r)$ generate D as well, therefore we can write

$$d_j = \sum_{k=1}^{r} g_{jk}\varphi(d_k)$$

for some $g_{jk} \in E_{\Delta_n}$. Choose a large enough $s \in \mathbb{N}$ such that $X_{\Delta_n}^{(q-1)s} g_{jk} \in E_{\Delta_n}^{+}$ for all $1 \leq j, k \leq r$. Since

$$X_{\Delta_n}^{-s} d_j = \sum_{k=1}^{r} X_{\Delta_n}^{(q-1)s} g_{jk}\varphi(X_{\Delta_n}^{-s} d_k),$$

it follows that $X_{\Delta_n}^{-s} d_1, \ldots, X_{\Delta_n}^{-s} d_r$ is a basis of D satisfying the conditions of Lemma 2.38. Hence

$$D^{++} \subseteq X_{\Delta_n}^{-s}(E_{\Delta_n}^{+} d_1 + \ldots + E_{\Delta_n}^{+} d_r)$$

and by Noetherianity, we get that D^{++} is a finitely generated $E_{\Delta_n}^{+}$-module, meaning it is a finitely generated module over $\mathcal{O}_{\Delta_n}^{+}$.

Case 2. Assume that we are in the general case. By Remark 2.30 the E_{Δ_n}-module D is stably free, so suppose that

$$D_1 \coloneqq D \oplus E_{\Delta_n}^{\oplus k} \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$$

is a finitely generated free module over E_{Δ_n} for some $k \in \mathbb{N}_{\geq 0}$. By Lemma 2.34 we also have that

$$D_1^{++} = D^{++} \oplus (E_{\Delta_n}^{\oplus k})^{++}$$

meaning that D^{++} can be regarded as an $E_{\Delta_n}^{+}$-submodule of D_1^{++}. Since D_1 is free over E_{Δ_n}, by Case 1 we know that D_1^{++} is a finitely generated $E_{\Delta_n}^{+}$-module. As D^{++} is a submodule of a Noetherian $E_{\Delta_n}^{+}$-module it must be finitely generated. $\square$

2.5.3 The proof

Another important observation is that the group $\Gamma_{\Delta_n,L}$ is not merely profinite, but it is also *strongly complete* as well, which means the following.

Proposition 2.40. *Every subgroup of $\Gamma_{\Delta_n,L}$ of finite index is open.*

Proof. Let H be a subgroup of $\Gamma_{\Delta_n,L}$ of finite index. For $k \in \mathbb{N}$, consider the open subgroup $U_L^{(k)} \coloneqq 1 + \pi^k \mathcal{O}_L$ of $\mathcal{O}_L^{\times}$. Choose $j \in \mathbb{N}$ large enough such that we have an isomorphism of topological groups

$$U_L^{(j)} \simeq \mathbb{Z}_p^{[L:\mathbb{Q}_p]}$$

where the latter is regarded as a topological group under addition (such a j exists by Proposition II.5.5 of [Neu99]). Let

$$U^{(j)}_{\Delta_n,L} := \prod_{i\in\Delta_n} U^{(j)}_{i,L} \subseteq \Gamma_{\Delta_n,L}$$

where $U^{(j)}_{i,L}$ denotes a copy of $U^{(j)}_{L}$ inside $\Gamma_{i,L} \simeq \mathcal{O}_L^\times$. Then $U^{(j)}_{\Delta_n,L}$ is an open subgroup of $\Gamma_{\Delta_n,L}$. It suffices to show that

$$H_1 := U^{(j)}_{\Delta_n,L} \cap H$$

is an open subgroup of $U^{(j)}_{\Delta_n,L}$. From now on, we identify $U^{(j)}_{\Delta_n,L}$ with $\mathbb{Z}_p^m$, where

$$m := n[L : \mathbb{Q}_p],$$

and H_1 with a subgroup of $\mathbb{Z}_p^m$. By our assumption about H, we know that H_1 is a finite index subgroup of $\mathbb{Z}_p^m$. Suppose that

$$[\mathbb{Z}_p^m : H_1] = s$$

for some $s \in \mathbb{N}$. Then $s\mathbb{Z}_p^m \subseteq H_1$. Writing $s = p^e u$ for $u, e \in \mathbb{N}$ with u coprime to p, we obtain that $s\mathbb{Z}_p^m = p^e\mathbb{Z}_p^m \subseteq H_1$. Therefore H_1 is open in $\mathbb{Z}_p^m$, as desired. $\square$

At this point, the proof of [Sch17] in the one variable case proceeds to show that the group action on the module is separately continuous when the module is annihilated by a power of π using the Colmez module as defined above. Furthermore, using the elementary divisor theorem for a finitely generated $\mathcal{A}_L$-module, Lemma 2.2.9 in [Sch17] shows that the module is locally compact for the weak topology. The author then quotes a theorem of Ellis in [Ell57] which says that the action of a locally compact group on a locally compact space is continuous if it is separately continuous.

In our multivariable setting however, the ring $\mathcal{A}_{\Delta_n}$ is no longer a PID if $n > 1$. Hence we cannot use the argument of Lemma 2.2.9 from [Sch17] to show that $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ is locally compact when $\pi^m D = 0$ for some $m \in \mathbb{N}_{\geq 1}$. Instead, we modify the argument of [Sch17] to prove directly that $\Gamma_{\Delta_n,L}$ acts continuously on D for the weak topology by using what we proved about the Colmez module D^{++}. Our proof also works for the one variable case and shows how one can avoid using that the module is locally compact as well as quoting the result of Ellis there.

Proposition 2.41. *Suppose that $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ such that $\pi^m D = 0$ for some $m > 0$. The map*

$$\Gamma_{\Delta_n,L} \times D \longrightarrow D$$

is continuous for the weak topology of D.

Proof. Using Proposition 2.39 we can consider a finite generating set $d_1, \ldots, d_r$ of D^{++} as an $\mathcal{O}^+_{\Delta_n}$-module. By Corollary 2.36 the elements $d_1, \ldots, d_r$ also a generate D over $\mathcal{O}_{\Delta_n,m}$ and by Remark 2.37 for $\ell \in \mathbb{Z}$ the sets

$$(\mathcal{O}_{\Delta_n,m})_\ell \cdot d_1 + \ldots + (\mathcal{O}_{\Delta_n,m})_\ell \cdot d_r = (\mathcal{O}_{\Delta_n,m})_\ell D^{++}$$

form a fundamental system of open neighbourhoods of zero in D for the weak topology. Let $\sigma \in \Gamma_{\Delta_n,L}$, $d \in D$ and

$$U \coloneqq \sigma(d) + (\mathcal{O}_{\Delta_n,m})_\ell D^{++}$$

a basic open neighbourhood of $\sigma(d)$ for some $\ell > 0$. First we show there exists an open subgroup H of $\Gamma_{\Delta_n,L}$ such that

$$H\sigma(d) \subseteq \sigma(d) + (\mathcal{O}_{\Delta_n,m})_\ell D^{++}. \tag{2.14}$$

By Corollary 2.36 we can assume that $d \in X_{\Delta_n}^{-k} D^{++}$ for some $k \in \mathbb{N}_{\geq 0}$ large enough. By Lemma 2.33 (iii) it follows that $\sigma(d) \in X_{\Delta_n}^{-k} D^{++}$ as well. Consider also the set

$$(\mathcal{O}_{\Delta_n,m})_{\ell,-k} \coloneqq \sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{Z}^n \\ i_1+\ldots+i_n=\ell \\ i_1,\ldots,i_n\geq -k}} X_1^{i_1} \ldots X_n^{i_n} \mathcal{O}^+_{\Delta_n,m} = X_{\Delta_n}^{-k} \mathcal{O}^+_{\Delta_n,m} \cap (\mathcal{O}_{\Delta_n,m})_\ell.$$

Then $(\mathcal{O}_{\Delta_n,m})_{\ell,-k} D^{++} \subseteq X_{\Delta_n}^{-k} D^{++}$ and by Lemma 2.33 (iii) both $(\mathcal{O}_{\Delta_n,m})_{\ell,-k} D^{++}$ and $X_{\Delta_n}^{-k} D^{++}$ are preserved by $\Gamma_{\Delta_n,L}$, therefore we have a natural group homomorphism

$$\psi : \Gamma_{\Delta_n,L} \to \mathrm{Aut}\left(X_{\Delta_n}^{-k} D^{++} / (\mathcal{O}_{\Delta_n,m})_{\ell,-k} D^{++}\right).$$

By Proposition 2.39 $X_{\Delta_n}^{-k} D^{++}$ is a finitely generated $\mathcal{O}^+_{\Delta_n,m}$-module and

$$(X_1, \ldots, X_n)^{\ell+nk} X_{\Delta_n}^{-k} D^{++} \subseteq (\mathcal{O}_{\Delta_n,m})_{\ell,-k} D^{++},$$

therefore $X_{\Delta_n}^{-k} D^{++} / (\mathcal{O}_{\Delta_n,m})_{\ell,-k} D^{++}$ is a finitely generated module over the ring

$$\mathcal{O}^+_{\Delta_n,m} / (X_1, \ldots, X_n)^{\ell+nk} \mathcal{O}^+_{\Delta_n,m}.$$

Since $\mathcal{O}^+_{\Delta_n,m} / (X_1, \ldots, X_n)^{\ell+nk} \mathcal{O}^+_{\Delta_n,m}$ is a finite set, it follows that $X_{\Delta_n}^{-k} D^{++} / (\mathcal{O}_{\Delta_n,m})_{\ell,-k} D^{++}$ and its group of automorphisms

$$\mathrm{Aut}\left(X_{\Delta_n}^{-k} D^{++} / (\mathcal{O}_{\Delta_n,m})_{\ell,-k} D^{++}\right)$$

have finite cardinality as well. Therefore $H \coloneqq \ker \psi$ is a finite index subgroup of $\Gamma_{\Delta_n,L}$. By Proposition 2.40 H is open in $\Gamma_{\Delta_n,L}$ and satisfies (2.14) because $(\mathcal{O}_{\Delta_n,m})_{\ell,-k} D^{++} \subseteq (\mathcal{O}_{\Delta_n,m})_\ell D^{++}$ and $\sigma(d) \in X_{\Delta_n}^{-k} D^{++}$. Therefore

$$\begin{aligned}
H\sigma\left(d + (\mathcal{O}_{\Delta_n,m})_\ell D^{++}\right) &= H\sigma(d) + H\sigma\left((\mathcal{O}_{\Delta_n,m})_\ell D^{++}\right) \\
&\subseteq \sigma(d) + (\mathcal{O}_{\Delta_n,m})_\ell D^{++} + H\sigma\left((\mathcal{O}_{\Delta_n,m})_\ell D^{++}\right) \\
&\subseteq \sigma(d) + (\mathcal{O}_{\Delta_n,m})_\ell D^{++} + \Gamma_{\Delta_n,L}\left((\mathcal{O}_{\Delta_n,m})_\ell D^{++}\right) \\
&\subseteq \sigma(d) + (\mathcal{O}_{\Delta_n,m})_\ell D^{++} + (\mathcal{O}_{\Delta_n,m})_\ell D^{++} \\
&= \sigma(d) + (\mathcal{O}_{\Delta_n,m})_\ell D^{++},
\end{aligned}$$

where the inclusion in the second line follows from (2.14) and the inclusion from the fourth line follows from Lemma 2.33 (iii). $\square$

Combining the results of Proposition 2.31 and Proposition 2.41, we obtain the result stated at the beginning of the section.

Theorem 2.42. *Suppose that $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. The map*

$$\Gamma_{\Delta_n,L} \times D \longrightarrow D$$

is continuous for the weak topology of D.

Chapter 3

The functors

In this chapter, we will define the functors between our two categories. For this, we first define a multivariable version of $\mathbb{C}_p^\flat$ in Section 3.1. Our new ring will be a perfect κ_L-algebra equipped with a $G_{\Delta_n,L}$-action and a norm for which it is complete. In Section 3.2 we explain how our new ring embeds the relevant rings of characteristic p. We also define a large ring of characteristic 0 in Section 3.3 and explain in Section 3.4 how it embeds $\mathcal{A}_{\Delta_n}$. In Section 3.5 we define a multivariable version of ring of integers of the maximal unramified extension of the field of fractions of $\mathcal{A}_L$ and explain its embedding into our large ring of characteristic 0. Our constructions of this chapter will later take care of the topological subtleties that arise when trying to prove that our functors are well defined. In Section 3.6, after computing some invariants, we introduce the functors between our categories.

3.1 The ring $\mathbb{C}_{p,\Delta_n}^\flat$

In this section we construct a multivariable version of $\mathbb{C}_p^\flat$. This new ring will be a perfect κ_L-algebra equipped with a continuous action from $G_{\Delta_n,L}$ and equipped with a norm for which it is complete.

We start by defining

$$\mathcal{O}_{\mathbb{C}_p^\flat,\Delta_n,\circ} := \bigotimes_{i\in\Delta_n,\kappa_L} \mathcal{O}_{\mathbb{C}_p^\flat}$$

and

$$\mathbb{C}_{p,\Delta_n,\circ}^\flat := \bigotimes_{i\in\Delta_n,\kappa_L} \mathbb{C}_p^\flat.$$

Lemma 3.1. *$\mathbb{C}_{p,\Delta_n,\circ}^\flat$ is a perfect κ_L-algebra.*

Proof. We want to show that the p-th power map on $\mathbb{C}_{p,\Delta_n,\circ}^\flat$ is bijective. The result of Chapter V § 15 Theorem 3 (d) of [Bou03] states that the tensor product of two reduced commutative algebras over a perfect field is reduced. Applying the result repeatedly, we obtain that $\mathbb{C}^\flat_{p,\Delta_n,\circ}$ is reduced. The p-th power map on $\mathbb{C}^\flat_{p,\Delta_n,\circ}$ is also surjective, since $\operatorname{char}\mathbb{C}^\flat_{p,\Delta_n,\circ} = p$ and the p-th power map is surjective on $\mathbb{C}^\flat_p$ and κ_L. □

Consider $|\cdot|_{\mathrm{prod}} : \mathbb{C}^\flat_{p,\Delta_n,\circ} \longrightarrow \mathbb{R}_{\geq 0}$ defined by

$$|x|_{\mathrm{prod}} := \inf\left\{\max_j\left\{\prod_{i\in\Delta_n}|x_{ij}|_\flat\right\} : x = \sum_{j=1}^r \bigotimes_{i\in\Delta_n} x_{ij} \text{ for } x_{ij}\in\mathbb{C}^\flat_p\right\}$$

for $x \in \mathbb{C}^\flat_{p,\Delta_n,\circ}$, where the infimum is taken over all the possible ways to write x as a sum $\sum_{j=1}^r \bigotimes_{i\in\Delta_n} x_{ij}$ with $x_{ij}\in\mathbb{C}^\flat_p$. It is clear that

(i) $|x|_{\mathrm{prod}} \geq 0$,

(ii) $|xy|_{\mathrm{prod}} \leq |x|_{\mathrm{prod}}|y|_{\mathrm{prod}}$,

(iii) $|x+y|_{\mathrm{prod}} \leq \max\{|x|_{\mathrm{prod}}, |y|_{\mathrm{prod}}\}$,

(iv) $|1|_{\mathrm{prod}} \leq 1$

for every $x, y \in \mathbb{C}^\flat_{p,\Delta_n,\circ}$. Therefore $|\cdot|_{\mathrm{prod}}$ is a semi-norm on $\mathbb{C}^\flat_{p,\Delta_n,\circ}$. As a matter of fact, by Proposition 17.4 of [Sch02], $|\cdot|_{\mathrm{prod}}$ is a norm.

Lemma 3.2. *We have that* $|x_1\otimes\ldots\otimes x_n|_{\mathrm{prod}} = \prod_{i\in\Delta_n}|x_i|_\flat$ *for* $x_i\in\mathbb{C}^\flat_p$.

Proof. It is clear by definition that

$$|x_1\otimes\ldots\otimes x_n|_{\mathrm{prod}} \leq \prod_{i\in\Delta_n}|x_i|_\flat.$$

Assume that $|x_1\otimes\ldots\otimes x_n|_{\mathrm{prod}} < \prod_{i\in\Delta_n}|x_i|_\flat$. Then we can write

$$x_1\otimes\ldots\otimes x_n = \sum_{j=1}^r \bigotimes_{i\in\Delta_n} y_{ij} \tag{3.1}$$

for some $y_{ij}\in\mathbb{C}^\flat_p$ such that

$$\max_{1\leq j\leq r}\left\{\prod_{i\in\Delta_n}|y_{ij}|_\flat\right\} < \prod_{i\in\Delta_n}|x_i|_\flat. \tag{3.2}$$

For $i \in \Delta_n$, let $c_i \in \mathcal{O}_{\mathbb{C}_p^\flat}$ be nonzero such that $\widetilde{x}_i := c_i x_i$ and $\widetilde{y}_{ij} := c_i y_{ij}$ are in $\mathcal{O}_{\mathbb{C}_p^\flat}$ for all $1 \leq j \leq r$. Multiply both sides of (3.1) by $c_1 \otimes \ldots \otimes c_n \in \mathcal{O}_{\mathbb{C}_p^\flat, \Delta_n, \circ}$. Then we have the identity

$$\widetilde{x}_1 \otimes \ldots \otimes \widetilde{x}_n = \sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} \widetilde{y}_{ij}$$

and since c_i are nonzero, by (3.2) we also have the inequality

$$\max_{1 \leq j \leq r} \left\{ \prod_{i \in \Delta_n} |\widetilde{y}_{ij}|_\flat \right\} < \prod_{i \in \Delta_n} |\widetilde{x}_i|_\flat.$$

By the density of $|\cdot|_\flat$ we can choose elements $x'_i \in \mathcal{O}_{\mathbb{C}_p^\flat}$ such that $|x'_i|_\flat < |\widetilde{x}_i|_\flat$ for all $i \in \Delta_n$ and still have that

$$\max_{1 \leq j \leq r} \left\{ \prod_{i \in \Delta_n} |\widetilde{y}_{ij}|_\flat \right\} < \prod_{i \in \Delta_n} |x'_i|_\flat.$$

Therefore, for each $1 \leq j \leq r$, there exists an index $i(j) \in \Delta_n$ such that

$$|\widetilde{y}_{i(j)j}|_\flat < |x'_{i(j)}|_\flat. \tag{3.3}$$

Consider the projection map

$$\mathrm{pr} : \mathcal{O}_{\mathbb{C}_p^\flat, \Delta_n, \circ} \to \bigotimes_{i \in \Delta_n, \kappa_L} \mathcal{O}_{\mathbb{C}_p^\flat} / x'_i \mathcal{O}_{\mathbb{C}_p^\flat}.$$

By (3.3) it follows that

$$\mathrm{pr}\left(\widetilde{x}_1 \otimes \ldots \otimes \widetilde{x}_n\right) = \mathrm{pr}\left(\sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} \widetilde{y}_{ij}\right) = 0.$$

On the other hand

$$\mathrm{pr}\left(\widetilde{x}_1 \otimes \ldots \otimes \widetilde{x}_n\right) = \left(\widetilde{x}_1 \bmod x'_1 \mathcal{O}_{\mathbb{C}_p^\flat}\right) \otimes \ldots \otimes \left(\widetilde{x}_n \bmod x'_n \mathcal{O}_{\mathbb{C}_p^\flat}\right) \neq 0$$

since $\widetilde{x}_i \bmod x'_i \mathcal{O}_{\mathbb{C}_p^\flat} \neq 0$ for all $i \in \Delta_n$. Therefore we get the desired contradiction. $\square$

Lemma 3.3. *For $x \in \mathbb{C}_{p, \Delta_n, \circ}^\flat$, we have that*

$$|x^p|_{\mathrm{prod}} = |x|_{\mathrm{prod}}^p.$$

Proof. Note that if x can be written as

$$x = \sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} y_{ij},$$

then we can write

$$x^p = \sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} y_{ij}^p$$

because $\mathrm{char}(\mathbb{C}^\flat_{p,\Delta_n,\circ}) = p$. Conversely, we know by Lemma 3.1 that every element of $\mathbb{C}^\flat_{p,\Delta_n,\circ}$ has a unique p-th root, therefore if we can write

$$x^p = \sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} z_{ij},$$

then we can write

$$x = \sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} z_{ij}^{1/p}.$$

Therefore we have a correspondence between the ways of writing x as a sum of pure tensors and writing x^p as a sum of pure tensors. The absolute value $|\cdot|_\flat$ is multiplicative on $\mathbb{C}^\flat_p$, therefore by definition it follows that

$$|x^p|_{\mathrm{prod}} = |x|^p_{\mathrm{prod}},$$

as desired. $\square$

Using the existing action of G_L on $\mathbb{C}^\flat_p$, we can define an action of $G_{\Delta_n,L}$ on $\mathbb{C}^\flat_{p,\Delta_n,\circ}$ by the formula

$$G_{\Delta_n,L} \times \mathbb{C}^\flat_{p,\Delta_n,\circ} \to \mathbb{C}^\flat_{p,\Delta_n,\circ}$$
$$\left(\prod_{i \in \Delta_n} \sigma_i\right) \times (x_1 \otimes \ldots \otimes x_n) \mapsto \sigma_1(x_1) \otimes \ldots \otimes \sigma_n(x_n).$$

Lemma 3.4. *Every element of $G_{\Delta_n,L}$ preserves the absolute value of $\mathbb{C}^\flat_{p,\Delta_n,\circ}$.*

Proof. Let $\sigma = \prod_{i \in \Delta_n} \sigma_i \in G_{\Delta_n,L}$. Note that if $x \in \mathbb{C}^\flat_{p,\Delta_n,\circ}$ and $\sigma(x)$ is written as

$$\sigma(x) = \sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} y_{ij},$$

then we can write

$$x = \sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} \sigma_i^{-1}(y_{ij})$$

and viceversa. Therefore σ induces a correspondence between the ways of writing x as a sum of pure tensors and writing $\sigma(x)$ as a sum of pure tensors. By Lemma 1.36 each σ_i preserves the absolute value on $\mathbb{C}^\flat_p$, therefore by definition it follows that $|x|_{\mathrm{prod}} = |\sigma(x)|_{\mathrm{prod}}$. $\square$

Lemma 3.5. *The group $G_{\Delta_n,L}$ acts continuously on $\mathbb{C}_{p,\Delta_n,\circ}^\flat$ for the topology induced by $|\cdot|_{\mathrm{prod}}$.*

Proof. Let $x \in \mathbb{C}_{p,\Delta_n,\circ}^\flat$ and $\sigma = \prod_{i\in\Delta_n} \sigma_i \in G_{\Delta_n,L}$. Fix an $\varepsilon \in \mathbb{R}_{>0}$ and write

$$x = \sum_{j=1}^{r} \bigotimes_{i\in\Delta_n} x_{ij}$$

for some $x_{ij} \in \mathbb{C}_p^\flat$. Let $\delta' := \max\{|x_{ij}|_\flat^{n-1} : i \in \Delta_n, 1 \le j \le r\}$ and

$$\delta := \frac{1}{\max\{1, \delta'\}}.$$

By Proposition 1.37, there exist open subgroups $\widetilde{G}_i \subseteq G_{i,L}$ such that

$$|h_i\sigma_i(x_{ij}) - \sigma_i(x_{ij})|_\flat < \varepsilon\delta \tag{3.4}$$

for all $h_i \in \widetilde{G}_i$, $i \in \Delta_n$ and $1 \le j \le r$. Then for $h = \prod_{i\in\Delta_n} h_i \in \prod_{i\in\Delta_n} \widetilde{G}_i$ we have that

$$h\sigma(x) - \sigma(x) = \sum_{j=1}^{r} \bigotimes_{i\in\Delta_n} h_i\sigma_i(x_{ij}) - \sum_{j=1}^{r} \bigotimes_{i\in\Delta_n} \sigma_i(x_{ij})$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{r} \Big(\sigma_1(x_{1j})\otimes\ldots\otimes\sigma_{i-1}(x_{(i-1)j})\otimes(h_i\sigma_i(x_{ij})-\sigma_i(x_{ij}))\otimes h_{i+1}\sigma_{i+1}(x_{(i+1)j})\ldots\otimes h_n\sigma_n(x_{nj})\Big). \tag{3.5}$$

Since every element of G_L preserves $|\cdot|_\flat$, by (3.4) we have that every summand of (3.5) has absolute value $|\cdot|_{\mathrm{prod}}$ less than ε. Therefore

$$|h\sigma(x) - \sigma(x)|_{\mathrm{prod}} < \varepsilon \tag{3.6}$$

for all $h \in \prod_{i\in\Delta_n} \widetilde{G}_i$. Let $y \in \mathbb{C}_{p,\Delta_n,\circ}^\flat$ such that $|x-y|_{\mathrm{prod}} < \varepsilon$. Then

$$\begin{aligned}
|h\sigma(y) - \sigma(x)|_{\mathrm{prod}} &= |h\sigma(y) - h\sigma(x) + h\sigma(x) - \sigma(x)|_{\mathrm{prod}} \\
&\le \max\{|h\sigma(y) - h\sigma(x)|_{\mathrm{prod}}, |h\sigma(x) - \sigma(x)|_{\mathrm{prod}}\} \\
&= \max\{|h\sigma(y-x)|_{\mathrm{prod}}, |h\sigma(x) - \sigma(x)|_{\mathrm{prod}}\} \\
&= \max\{|y-x|_{\mathrm{prod}}, |h\sigma(x) - \sigma(x)|_{\mathrm{prod}}\} \\
&< \varepsilon,
\end{aligned}$$

where in the equality in the fourth line we used Lemma 3.4 and in the inequality in the last line we used (3.6) and the condition on y. □

Let $\mathbb{C}^\flat_{p,\Delta_n}$ denote the completion of $\mathbb{C}^\flat_{p,\Delta_n,\circ}$ with respect to $|\cdot|_{\mathrm{prod}}$, which we can regard as the set of equivalence classes of Cauchy sequences of elements of $\mathbb{C}^\flat_{p,\Delta_n,\circ}$. Then $\mathbb{C}^\flat_{p,\Delta_n}$ naturally inherits the structure of a κ_L-algebra and is our desired multivariable analog of $\mathbb{C}^\flat_p$. We now show that $\mathbb{C}^\flat_{p,\Delta_n}$ also inherits some other aforementioned properties from $\mathbb{C}^\flat_{p,\Delta_n,\circ}$. We extend $|\cdot|_{\mathrm{prod}}$ to $\mathbb{C}^\flat_{p,\Delta_n}$ by defining

$$|y|_{\mathrm{prod}} \coloneqq \lim_{j\to\infty} |y_j|_{\mathrm{prod}},$$

for $y \in \mathbb{C}^\flat_{p,\Delta_n}$ represented as a Cauchy sequence $(y_0, y_1, \ldots)$ of elements in $\mathbb{C}^\flat_{p,\Delta_n,\circ}$.

Proposition 3.6. $\mathbb{C}^\flat_{p,\Delta_n}$ *is a perfect* κ_L*-algebra.*

Proof. We want to show that the p-th power map on $\mathbb{C}^\flat_{p,\Delta_n}$ is bijective. To show that the p-th power map on $\mathbb{C}^\flat_{p,\Delta_n}$ is injective, let y be a nonzero element in $\mathbb{C}^\flat_{p,\Delta_n}$. If we write

$$y = (y_0, y_1, \ldots)$$

as a Cauchy sequence of elements in $\mathbb{C}^\flat_{p,\Delta_n,\circ}$, then we know that $(y_0, y_1, \ldots)$ is not a null sequence. Therefore, by Lemma 3.3 $(y_0^p, y_1^p, \ldots)$ is not a null sequence as well, implying that $y^p \neq 0$.

Now we show that the p-th power map on $\mathbb{C}^\flat_{p,\Delta_n}$ is also surjective. Let $y \in \mathbb{C}^\flat_{p,\Delta_n}$ which is represented by a Cauchy sequence

$$y = (y_0, y_1, \ldots)$$

of elements y_j in $\mathbb{C}^\flat_{p,\Delta_n,\circ}$. By Lemma 3.1 there exist unique p-th roots $y_j^{1/p}$ of y_j and by Lemma 3.3 it follows that

$$\beta = (y_0^{1/p}, y_1^{1/p}, \ldots)$$

is also a Cauchy sequence. Clearly $\beta^p = y$ and the conclusion follows. □

Lemma 3.7. *Let* $x_1, \ldots, x_n \in \mathbb{C}^\flat_p$ *and* $y \in \mathbb{C}^\flat_{p,\Delta_n}$. *Then*

$$|y(x_1 \otimes \ldots \otimes x_n)|_{\mathrm{prod}} = |y|_{\mathrm{prod}} \prod_{i\in\Delta_n} |x_i|_\flat.$$

Proof. If any of the x_i equals zero, the claimed equality is obvious, so assume that the x_i are nonzero. We first show that this holds for $y \in \mathbb{C}^\flat_{p,\Delta_n,\circ}$. Indeed, if we write

$$y = \sum_{j=1}^{r} \bigotimes_{i\in\Delta_n} y_{ij},$$

for some $y_{ij} \in \mathbb{C}^\flat_p$, then we can write

$$y(x_1 \otimes \ldots \otimes x_n) = \sum_{j=1}^{r} \bigotimes_{i\in\Delta_n} y_{ij} x_i.$$

Conversely, if we write

$$y(x_1 \otimes \ldots \otimes x_n) = \sum_{j=1}^{s} \bigotimes_{i \in \Delta_n} z_{ij},$$

for some $z_{ij} \in \mathbb{C}_p^\flat$, then we can write

$$y = \sum_{j=1}^{s} \bigotimes_{i \in \Delta_n} z_{ij} x_i^{-1}.$$

Therefore we have a correspondence between the ways of writing y as a sum of pure tensors and writing $y(x_1 \otimes \ldots \otimes x_n)$ as a sum of pure tensors. By definition it follows that

$$|y(x_1 \otimes \ldots \otimes x_n)|_{\mathrm{prod}} = |y|_{\mathrm{prod}} \prod_{i \in \Delta_n} |x_i|_\flat.$$

Now suppose that $y \in \mathbb{C}_{p,\Delta_n}^\flat$ and write $y = (y_0, y_1, \ldots)$ as a Cauchy sequence of elements in $\mathbb{C}_{p,\Delta_n,\circ}^\flat$. Then

$$y(x_1 \otimes \ldots \otimes x_n) = (y_0(x_1 \otimes \ldots \otimes x_n), y_1(x_1 \otimes \ldots \otimes x_n), \ldots)$$

and we obtain that

$$\begin{aligned} |y(x_1 \otimes \ldots \otimes x_n)|_{\mathrm{prod}} &= \lim_{j \to \infty} |y_j(x_1 \otimes \ldots \otimes x_n)|_{\mathrm{prod}} \\ &= \lim_{j \to \infty} |y_j|_{\mathrm{prod}} \prod_{i \in \Delta_n} |x_i|_\flat \\ &= \prod_{i \in \Delta_n} |x_i|_\flat \lim_{j \to \infty} |y_j|_{\mathrm{prod}} \\ &= |y|_{\mathrm{prod}} \prod_{i \in \Delta_n} |x_i|_\flat, \end{aligned}$$

as desired. $\square$

By Lemma 3.4, it follows that every element of $G_{\Delta_n,L}$ preserves Cauchy sequences in $\mathbb{C}_{p,\Delta_n,\circ}^\flat$, therefore $G_{\Delta_n,L}$ acts naturally on $\mathbb{C}_{p,\Delta_n}^\flat$.

Lemma 3.8. *Every element of $G_{\Delta_n,L}$ preserves the absolute value of $\mathbb{C}_{p,\Delta_n}^\flat$.*

Proof. Let y be an element of $\mathbb{C}_{p,\Delta_n}^\flat$ and $\sigma \in G_{\Delta_n,L}$. Then y can be represented by a Cauchy sequence $(y_0, y_1, \ldots)$ of elements in $\mathbb{C}_{p,\Delta_n,\circ}^\flat$. Then by Lemma 3.4 we have that

$$|\sigma(y)|_{\mathrm{prod}} = \lim_{j \to \infty} |\sigma(y_j)|_{\mathrm{prod}} = \lim_{j \to \infty} |y_j|_{\mathrm{prod}} = |y|_{\mathrm{prod}}.$$

$\square$

Proposition 3.9. *The group $G_{\Delta_n,L}$ acts continuously on $\mathbb{C}_{p,\Delta_n}^\flat$.*

Proof. Let $x \in \mathbb{C}^\flat_{p,\Delta_n}, \sigma = \prod_{i\in\Delta_n} \sigma_i \in G_{\Delta_n,L}$ and fix an $\varepsilon \in \mathbb{R}_{>0}$. Suppose that $\widetilde{x} \in \mathbb{C}^\flat_{p,\Delta_n,\circ}$ is such that

$$|x - \widetilde{x}|_{\text{prod}} < \varepsilon.$$

By Lemma 3.5 there exists an open subgroup $H \subseteq G_{\Delta_n,L}$ such that

$$|h\sigma(\widetilde{x}) - \sigma(\widetilde{x})|_{\text{prod}} < \varepsilon \tag{3.7}$$

for all $h \in H$. Let $y \in \mathbb{C}^\flat_{p,\Delta_n}$ such that $|x - y|_{\text{prod}} < \varepsilon$. We then have that

$$h\sigma(y) - \sigma(x) = h\sigma(y) - h\sigma(x) + h\sigma(x) - h\sigma(\widetilde{x}) + h\sigma(\widetilde{x}) - \sigma(\widetilde{x}) + \sigma(\widetilde{x}) - \sigma(x).$$

Then by Lemma 3.8 we have that $|h\sigma(y) - h\sigma(x)|_{\text{prod}} < \varepsilon, |h\sigma(x) - h\sigma(\widetilde{x})|_{\text{prod}} < \varepsilon$ and $|\sigma(\widetilde{x}) - \sigma(x)|_{\text{prod}} < \varepsilon$. Together with (3.7) we conclude that

$$|h\sigma(y) - \sigma(x)|_{\text{prod}} < \varepsilon$$

for all $h \in H$ and $y \in \mathbb{C}^\flat_{p,\Delta_n}$ with the aforementioned property, as desired. $\square$

3.2 The ring $E^{\text{sep}}_{\Delta_n}$ and its embedding into $\mathbb{C}^\flat_{p,\Delta_n}$

In this section we define a multivariable analog of $\kappa_L(\!(X)\!)^{\text{sep}}$ and embed it into $\mathbb{C}^\flat_{p,\Delta_n}$. For $i \in \Delta_n$, we let $E_i := \kappa_L(\!(X_i)\!)$ and $E_i^+ := \kappa_L[\![X_i]\!]$. We also let E_i^{sep} be a separable closure of E_i and $E_i^{\text{sep}+}$ denote its ring of integers. We define

$$E^{\text{sep}}_{\Delta_n,\circ} := \bigotimes_{i\in\Delta_n,\kappa_L} E_i^{\text{sep}},$$

$$E^{\text{sep}+}_{\Delta_n,\circ} := \bigotimes_{i\in\Delta_n,\kappa_L} E_i^{\text{sep}+},$$

$$E_{\Delta_n,\circ} := \bigotimes_{i\in\Delta_n,\kappa_L} E_i$$

and

$$E^+_{\Delta_n,\circ} := \bigotimes_{i\in\Delta_n,\kappa_L} E_i^+.$$

Our multivariable analog of $\kappa_L(\!(X)\!)^{\text{sep}}$ is the ring

$$E^{\text{sep}}_{\Delta_n} := E^{\text{sep}}_{\Delta_n,\circ} \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n}.$$

We also keep track of the following integral version of $E^{\text{sep}}_{\Delta_n}$, which define to be

$$E^{\text{sep}+}_{\Delta_n} := E^{\text{sep}+}_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} E^+_{\Delta_n}.$$

We now provide alternative descriptions of $E^{\text{sep}}_{\Delta_n}$ and $E^{\text{sep}+}_{\Delta_n}$ which we will use for computational purposes. One of them expresses these rings as rearranged tensor products and it is given in the following lemma.

Lemma 3.10. *(i) The map*

$$\begin{aligned} E_{\Delta_n}^{\mathrm{sep}} &\to E_1^{\mathrm{sep}} \otimes_{E_1} (E_2^{\mathrm{sep}} \otimes_{E_2} (\ldots \otimes (E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n}))) \\ (e_1 \otimes \ldots \otimes e_n) \otimes f &\mapsto e_1 \otimes (e_2 \otimes \ldots \otimes (e_n \otimes f)) \end{aligned}$$

is an isomorphism, where $f \in E_{\Delta_n}$ and $e_i \in E_i^{\mathrm{sep}}$ for $i \in \Delta_n$.

(ii) The map

$$\begin{aligned} E_{\Delta_n}^{\mathrm{sep}+} &\to E_1^{\mathrm{sep}+} \otimes_{E_1^+} \left(E_2^{\mathrm{sep}+} \otimes_{E_2^+} \left(\ldots \otimes \left(E_n^{\mathrm{sep}+} \otimes_{E_n^+} E_{\Delta_n}^+\right)\right)\right) \\ (e_1 \otimes \ldots \otimes e_n) \otimes f &\mapsto e_1 \otimes (e_2 \otimes \ldots \otimes (e_n \otimes f)) \end{aligned}$$

is an isomorphism, where $f \in E_{\Delta_n}^+$ and $e_i \in E_i^{\mathrm{sep}+}$ for $i \in \Delta_n$.

Proof. (i) Rearranging the tensor products gives us the isomorphisms

$$\begin{aligned} E_{\Delta_n}^{\mathrm{sep}} = E_{\Delta_n,\circ}^{\mathrm{sep}} \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n} &\simeq \left(\bigotimes_{i \in \Delta_n, \kappa_L} (E_i^{\mathrm{sep}} \otimes_{E_i} E_i) \right) \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n} \\ &\simeq \left(E_1^{\mathrm{sep}} \otimes_{E_1} (\ldots (E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n,\circ})) \right) \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n} \\ &\simeq E_1^{\mathrm{sep}} \otimes_{E_1} (\ldots (E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n})), \end{aligned}$$

whose composition is the map in the claim.

(ii) The proof is entirely analogous to that of part (i).

□

Remark 3.11. The same proof shows that we also have the isomorphisms

$$E_{\Delta_n}^{\mathrm{sep}} \simeq E_{\alpha_1}^{\mathrm{sep}} \otimes_{E_{\alpha_1}} \left(E_{\alpha_2}^{\mathrm{sep}} \otimes_{E_{\alpha_2}} \left(\ldots \otimes \left(E_{\alpha_n}^{\mathrm{sep}} \otimes_{E_{\alpha_n}} E_{\Delta_n}\right)\right)\right)$$

and

$$E_{\Delta_n}^{\mathrm{sep}+} \simeq E_{\alpha_1}^{\mathrm{sep}+} \otimes_{E_{\alpha_1}^+} \left(E_{\alpha_2}^{\mathrm{sep}+} \otimes_{E_{\alpha_2}^+} \left(\ldots \otimes \left(E_{\alpha_n}^{\mathrm{sep}+} \otimes_{E_{\alpha_n}^+} E_{\Delta_n}^+\right)\right)\right)$$

where $(\alpha_1, \ldots, \alpha_n)$ is an arbitrary permutation of the set Δ_n.

For another alternative description of $E_{\Delta_n}^{\mathrm{sep}}$, we observe that the commutativity of tensor products with colimits shows that

$$E_{\Delta_n}^{\mathrm{sep}} \simeq \varinjlim_{\substack{E_i \subseteq E_i' \subseteq E_i^{\mathrm{sep}} \\ [E_i':E_i]<\infty,\ i \in \Delta_n}} (E_1' \otimes_{\kappa_L} \ldots \otimes_{\kappa_L} E_n') \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n}$$

and

$$E_{\Delta_n}^{\mathrm{sep}+} \simeq \varinjlim_{\substack{E_i \subseteq E_i' \subseteq E_i^{\mathrm{sep}} \\ [E_i':E_i]<\infty,\ i \in \Delta_n}} \left(E_1'^+ \otimes_{\kappa_L} \ldots \otimes_{\kappa_L} E_n'^+\right) \otimes_{E_{\Delta_n,\circ}^+} E_{\Delta_n}^+$$

where the colimits are taken over all the collections of finite separable extensions $(E'_i/E_i)_{i\in\Delta_n}$ with E'^+_i being the ring of integers of E'_i. We denote

$$E'_{\Delta_n,\circ} := \bigotimes_{i\in\Delta_n,\kappa_L} E'_i,$$

$$E'^+_{\Delta_n,\circ} := \bigotimes_{i\in\Delta_n,\kappa_L} E'^+_i,$$

$$E'_{\Delta_n} := E'_{\Delta_n,\circ} \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n}$$

and

$$E'^+_{\Delta_n} := E'^+_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} E^+_{\Delta_n}.$$

Lemma 3.12. *(i) $E^{\mathrm{sep}}_{\Delta_n}$ is a flat E_{Δ_n}-module.*

(ii) $E^{\mathrm{sep}\,+}_{\Delta_n}$ is a flat $E^+_{\Delta_n}$-module.

(iii) The natural map

$$\begin{aligned} E_{\Delta_n} &\hookrightarrow E^{\mathrm{sep}}_{\Delta_n} \\ e &\mapsto (1\otimes\ldots\otimes 1)\otimes e \end{aligned}$$

is an embedding.

(iv) The natural map

$$\begin{aligned} E'_{\Delta_n} &\hookrightarrow E^{\mathrm{sep}}_{\Delta_n} \\ (e_1\otimes\ldots\otimes e_n)\otimes e &\mapsto (e_1\otimes\ldots\otimes e_n)\otimes e \end{aligned}$$

is an embedding, where $e_i \in E'_i$ for $i\in\Delta_n$ and $e\in E_{\Delta_n}$.

Proof. Each E'_i is a finite free E_i-vector space and admits a basis over E_i containing the element 1, therefore $E'_{\Delta_n,\circ}$ is a finite free $E_{\Delta_n,\circ}$-module admitting a basis containing the element $1\otimes\ldots\otimes 1$. Hence E'_{Δ_n} is also a finite free E_{Δ_n}-module admitting a basis containing the element $(1\otimes\ldots\otimes 1)\otimes 1$. In particular, E'_{Δ_n} is a flat E_{Δ_n}-module and a colimit argument shows the claim in (i). Each E'^+_i is also a finite free E^+_i-module, therefore an analogous argument shows that (ii) holds as well. The fact that E'_{Δ_n} is a finite free E_{Δ_n}-module admitting a basis containing the element $(1\otimes\ldots\otimes 1)\otimes 1$ also shows that the natural map

$$\begin{aligned} E_{\Delta_n} &\hookrightarrow E'_{\Delta_n} \\ e &\mapsto (1\otimes\ldots\otimes 1)\otimes e \end{aligned}$$

is an embedding. Therefore the result of (iii) follows using a colimit argument. Considering another collection of finite separable extensions $(E''_i/E'_i)_{i\in\Delta_n}$, a similar argument shows that E''_{Δ_n} is a finite free E'_{Δ_n}-module admitting a basis containing the element $(1\otimes\ldots\otimes 1)\otimes 1$. Therefore the map

$$\begin{aligned} E'_{\Delta_n} &\hookrightarrow E''_{\Delta_n} \\ (e_1\otimes\ldots\otimes e_n)\otimes e &\mapsto (e_1\otimes\ldots\otimes e_n)\otimes e \end{aligned}$$

is an embedding and a colimit argument proves (iv). $\square$

Lemma 3.13. *(i) The map*

$$\begin{aligned} E'^{+}_{\Delta_n} &\to E'_{\Delta_n} \\ (e_1 \otimes \ldots \otimes e_n) \otimes f &\mapsto (e_1 \otimes \ldots \otimes e_n) \otimes f \end{aligned}$$

is injective, where $f \in E^+_{\Delta_n}$ and $e_i \in E'^+_i$ for $i \in \Delta_n$.

(ii) There is an isomorphism $E'_{\Delta_n} \simeq E'^{+}_{\Delta_n}[X^{-1}_{\Delta_n}]$.

(iii) The map

$$\begin{aligned} E'_{\Delta_n} &\to E'_1 \otimes_{E_1} (E'_2 \otimes_{E_2} (\ldots \otimes (E'_n \otimes_{E_n} E_{\Delta_n}))) \\ (e_1 \otimes \ldots \otimes e_n) \otimes f &\mapsto e_1 \otimes (e_2 \otimes \ldots \otimes (e_n \otimes f)) \end{aligned}$$

is an isomorphism, where $f \in E_{\Delta_n}$ and $e_i \in E'_i$ for $i \in \Delta_n$.

Proof. (i) One shows that the map is well defined by checking the linearity conditions in every component and using the universal property of the tensor product. E'^+_i is a finite free E^+_i-module, therefore $E'^{+}_{\Delta_n,\circ}$ is a finite free $E^+_{\Delta_n,\circ}$-module, hence applying $E'^{+}_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} -$ to the inclusion $E^+_{\Delta_n} \subseteq E_{\Delta_n}$ gives us an injective map

$$E'^{+}_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} E^+_{\Delta_n} \to E'^{+}_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} E_{\Delta_n} \tag{3.8}$$

by flatness. Moreover, E'^+_i and E'_i admit a common basis over E^+_i and E_i, respectively, for $i \in \Delta_n$. Therefore $E'^{+}_{\Delta_n,\circ}$ and $E'_{\Delta_n,\circ}$ admit a common basis over $E^+_{\Delta_n,\circ}$ and $E_{\Delta_n,\circ}$, respectively. This allows us to construct an additive bijection

$$E'^{+}_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} E_{\Delta_n} \simeq E'_{\Delta_n,\circ} \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n} \tag{3.9}$$

whose precomposition by (3.8) gives the desired map and the conclusion follows.

(ii) It is clear that we have the isomorphisms

$$\begin{aligned} E'^{+}_{\Delta_n}[X^{-1}_{\Delta_n}] &\simeq E'^{+}_{\Delta_n} \otimes_{E^+_{\Delta_n}} E^+_{\Delta_n}[X^{-1}_{\Delta_n}] \simeq (E'^{+}_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} E^+_{\Delta_n}) \otimes_{E^+_{\Delta_n}} E_{\Delta_n} \\ &\simeq E'^{+}_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} (E^+_{\Delta_n} \otimes_{E^+_{\Delta_n}} E_{\Delta_n}) \simeq E'^{+}_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} E_{\Delta_n}. \end{aligned}$$

By (3.9) we have an isomorphism

$$E'^{+}_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} E_{\Delta_n} \simeq E'_{\Delta_n,\circ} \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n} = E'_{\Delta_n}$$

which concludes our proof.

(iii) The proof is entirely analogous to that of Lemma 3.10. □

Corollary 3.14. *(i) The map*

$$\begin{aligned} E^{\mathrm{sep}+}_{\Delta_n} &\to E^{\mathrm{sep}}_{\Delta_n} \\ (e_1 \otimes \ldots \otimes e_n) \otimes f &\mapsto (e_1 \otimes \ldots \otimes e_n) \otimes f \end{aligned}$$

is injective, where $f \in E^+_{\Delta_n}$ and $e_i \in E^{\mathrm{sep}+}_i$ for $i \in \Delta_n$.

(ii) There is an isomorphism $E_{\Delta_n}^{\mathrm{sep}} \simeq E_{\Delta_n}^{\mathrm{sep}+}[X_{\Delta_n}^{-1}]$.

Proof. (i) This follows immediately from Lemma 3.13 (i) after taking colimits.

(ii) Localization and colimits commute, therefore the conclusion follows by part (ii) of Lemma 3.13. □

In the following, we explain how to regard each ring $E_{\Delta_n}'^{+}$ as a completed tensor product. For each $i \in \Delta_n$, fix a finite separable extension E_i' of E_i and write $E_i' = \mathbb{F}_{q_i}(\!(X_i')\!)$ where X_i' is a uniformizer of E_i' and q_i is a power of q. Then $E_i'^{+} = \mathbb{F}_{q_i}[\![X_i']\!]$.

We first prove a more general lemma about tensor products of power series rings.

Lemma 3.15. *Let k be a field, $k_1, \dots, k_n$ be extension fields of k and $t_1, \dots, t_n$ a collection of free variables.*

(i) The natural map

$$\psi : k_1[\![t_1]\!] \otimes_k \dots \otimes_k k_n[\![t_n]\!] \longrightarrow (k_1 \otimes_k \dots \otimes_k k_n)[\![t_1, \dots, t_n]\!]$$
$$\left(\sum_{j_1=0}^{\infty} c_{1j_1} t_1^{j_1}\right) \otimes \dots \otimes \left(\sum_{j_n=0}^{\infty} c_{nj_n} t_n^{j_n}\right) \longmapsto \sum_{(j_1,\dots,j_n)\in\mathbb{N}^n} (c_{1j_1} \otimes \dots \otimes c_{nj_n})\, t_1^{j_1} \dots t_n^{j_n}$$

is an embedding.

(ii) For $m \in \mathbb{N}_{\geq 1}$, we have that

$$\psi^{-1}\left((t_1, \dots, t_n)^m (k_1 \otimes_k \dots \otimes_k k_n)[\![t_1, \dots, t_n]\!]\right) = (\widetilde{t}_1, \dots, \widetilde{t}_n)^m (k_1[\![t_1]\!] \otimes_k \dots \otimes_k k_n[\![t_n]\!])$$

where $\widetilde{t}_i$ denotes the pure tensor in $k_1[\![t_1]\!] \otimes_k \dots \otimes_k k_n[\![t_n]\!]$ whose i-th component is t_i and other components are 1.

(iii) We have an isomorphism of k-algebras

$$(k_1 \otimes_k \dots \otimes_k k_n)[\![t_1, \dots, t_n]\!] \xrightarrow{\simeq} \varprojlim_{m\geq 1} k_1[\![t_1]\!] \otimes_k \dots \otimes_k k_n[\![t_n]\!]/(\widetilde{t}_1, \dots, \widetilde{t}_n)^m.$$

Proof. (i) *Step 1:* We first show that if A and B are commutative algebras over k and t is a free variable, then the natural map

$$\psi_1 : A \otimes_k B[\![t]\!] \longrightarrow (A \otimes_k B)[\![t]\!]$$
$$\sum_{i=1}^{m} a_i \otimes \left(\sum_{j=0}^{\infty} b_{ij} t^j\right) \longmapsto \sum_{j=0}^{\infty} \left(\sum_{i=1}^{m} a_i \otimes b_{ij}\right) t^j \tag{3.10}$$

is injective. Let y be a nonzero element in $A \otimes_k B[\![t]\!]$. Suppose that

$$y = \sum_{i=1}^{m} a_i \otimes \left(\sum_{j=0}^{\infty} b_{ij} t^j \right)$$

is an expression for y where m takes the smallest possible value. Since y is nonzero, we have that $m \geq 1$, and the minimality of m shows that $\{a_i\}_{1 \leq i \leq m}$ is a k-linearly independent set. If $\psi_1(y) = 0$, then

$$\sum_{i=1}^{m} a_i \otimes b_{ij} = 0$$

for each $j \geq 0$. Due to the linear independence of the a_i, it follows that $b_{ij} = 0$ for all i and j. Therefore $y = 0$ and we obtain a contradiction.

Step 2: We will prove the result by induction on n. For $n = 1$ the statement is trivial, so assume that $n \geq 2$. For the induction step, it suffices to show that the natural map

$$\psi_2 : (k_1 \otimes_k \ldots \otimes_k k_{n-1})[\![t_1, \ldots, t_{n-1}]\!] \otimes_k k_n[\![t_n]\!] \hookrightarrow (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!]$$

defined in an analogous way as ψ and ψ_1 is injective. Applying repeatedly the result of Step 1 we obtain a composition of embeddings

$$\begin{aligned}(k_1 \otimes_k \ldots \otimes_k k_{n-1})[\![t_1, \ldots, t_{n-1}]\!] \otimes_k k_n[\![t_n]\!] &\hookrightarrow ((k_1 \otimes_k \ldots \otimes_k k_{n-1})[\![t_1, \ldots t_{n-1}]\!] \otimes_k k_n) [\![t_n]\!] \\ &\hookrightarrow ((k_1 \otimes_k \ldots \otimes_k k_{n-1})[\![t_1, \ldots t_{n-2}]\!] \otimes_k k_n) [\![t_{n-1}, t_n]\!] \\ &\hookrightarrow \ldots \\ &\hookrightarrow (k_1 \otimes_k \ldots \otimes_k k_n) [\![t_1, \ldots t_n]\!]\end{aligned}$$

which coincides with ψ_2 and the conclusion follows.

(ii) It is clear that

$$\psi^{-1} \left((t_1, \ldots, t_n)^m (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!]\right) \supseteq (\widetilde{t}_1, \ldots, \widetilde{t}_n)^m (k_1[\![t_1]\!] \otimes_k \ldots \otimes_k k_n[\![t_n]\!])$$

holds. To show the other containment, let

$$f \in \psi^{-1} \left((t_1, \ldots, t_n)^m (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!]\right).$$

Write

$$f = \sum_{\ell=1}^{r} \bigotimes_{i \in \Delta_n} \left(\sum_{j=0}^{\infty} c_{ij\ell} t_i^j \right)$$

for some $c_{ij\ell} \in k_i$. Then isolating in every component the powers of t_i larger or equal than m, we obtain that

$$f = \sum_{\ell=1}^{r} \bigotimes_{i \in \Delta_n} \left(\sum_{j=0}^{m-1} c_{ij\ell} t_i^j \right) + g$$

for some

$$g \in (\widetilde{t}_1^m, \ldots, \widetilde{t}_n^m)(k_1[\![t_1]\!] \otimes_k \ldots \otimes_k k_n[\![t_n]\!]) \subseteq (\widetilde{t}_1, \ldots, \widetilde{t}_n)^m(k_1[\![t_1]\!] \otimes_k \ldots \otimes_k k_n[\![t_n]\!]).$$

In particular, by the proven containment we have that

$$\psi\left(\sum_{\ell=1}^{r} \bigotimes_{i\in\Delta_n} \left(\sum_{j=0}^{m-1} c_{ij\ell} t_i^j\right)\right) \in (t_1, \ldots, t_n)^m (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!].$$

This is equivalent to

$$\sum_{\substack{(j_1,\ldots,j_n)\in\mathbb{N}^n \\ j_1,\ldots,j_n\leq m-1}} \left(\sum_{\ell=1}^{r} \bigotimes_{i\in\Delta_n} c_{ij_i\ell}\right) t_1^{j_1} \ldots t_n^{j_n} \in (t_1, \ldots, t_n)^m (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!].$$

The latter means that $\sum_{\ell=1}^{r} \bigotimes_{i\in\Delta_n} c_{ij_i\ell} = 0$ whenever $j_1, \ldots, j_n \leq m-1$ and $j_1+\ldots+j_n \leq m-1$. Therefore

$$\sum_{\ell=1}^{r} \bigotimes_{i\in\Delta_n} \left(\sum_{j=0}^{m-1} c_{ij\ell} t_i^j\right) \in (\widetilde{t}_1, \ldots, \widetilde{t}_n)^m (k_1[\![t_1]\!] \otimes_k \ldots \otimes_k k_n[\![t_n]\!])$$

implying that $f \in (\widetilde{t}_1, \ldots, \widetilde{t}_n)^m(k_1[\![t_1]\!] \otimes_k \ldots \otimes_k k_n[\![t_n]\!])$, as desired.

(iii) By part (i) and because vector spaces are flat over their coefficient field we have that all of the natural maps

$$(k_1 \otimes_k \ldots \otimes_k k_n)[t_1, \ldots, t_n] \simeq \bigotimes_{i\in\Delta_n,k} k_i[t_i] \hookrightarrow \bigotimes_{i\in\Delta_n,k} k_i[\![t_i]\!] \hookrightarrow (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!]$$

are injective. By part (ii), this induces the injective maps

$$\begin{aligned}(k_1 \otimes_k \ldots \otimes_k k_n)[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^m &\hookrightarrow \left(\bigotimes_{i\in\Delta_n,k} k_i[\![t_i]\!]\right)/(\widetilde{t}_1, \ldots, \widetilde{t}_n)^m \\ &\hookrightarrow (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!]/(t_1, \ldots, t_n)^m\end{aligned}$$

for every $m \geq 1$. Since

$$\begin{aligned}&\varprojlim_m (k_1 \otimes_k \ldots \otimes_k k_n)[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^m \simeq (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!] \\ &\simeq \varprojlim_m (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!]/(t_1, \ldots, t_n)^m\end{aligned}$$

it follows that

$$\varprojlim_m k_1[\![t_1]\!] \otimes_k \ldots \otimes_k k_n[\![t_n]\!]/(\widetilde{t}_1, \ldots, \widetilde{t}_n)^m \simeq (k_1 \otimes_k \ldots \otimes_k k_n)[\![t_1, \ldots, t_n]\!],$$

as desired. □

Specializing the results of Lemma 3.15, we obtain the following.

Corollary 3.16. *(i) We have an isomorphism of κ_L-algebras*

$$E^+_{\Delta_n} \simeq \varprojlim_{m\geq 1} E^+_{\Delta_n,\circ}/(\widetilde{X}_1,\ldots,\widetilde{X}_n)^m E^+_{\Delta_n,\circ}$$

where $\widetilde{X}_i$ denotes the pure tensor in $E^+_{\Delta_n,\circ}$ whose i-th component is X_i and the other components are 1.

(ii) We have an isomorphism of κ_L-algebras

$$E'^+_{\Delta_n} \simeq (\mathbb{F}_{q_1} \otimes_{\kappa_L} \ldots \otimes_{\kappa_L} \mathbb{F}_{q_n}) [\![X'_1,\ldots,X'_n]\!].$$

(iii) The natural map

$$\mathbb{F}_{q_1}[\![X'_1]\!] \otimes_{\kappa_L} \ldots \otimes_{\kappa_L} \mathbb{F}_{q_n}[\![X'_n]\!] \longrightarrow (\mathbb{F}_{q_1} \otimes_{\kappa_L} \ldots \otimes_{\kappa_L} \mathbb{F}_{q_n}) [\![X'_1,\ldots,X'_n]\!]$$

is an embedding.

Proof. (i) This follows directly from Lemma 3.15 (iii) applied for $k = k_1 = \ldots = k_n = \kappa_L$.

(ii) Let $\widetilde{X}'_i$ denote the pure tensor in $E'^+_{\Delta_n,\circ}$ whose i-th component is X'_i and the other components are 1. On the one hand, by Lemma 3.15 (iii) applied for $k = \kappa_L$ and $k_i = \mathbb{F}_{q_i}$ when $i \in \Delta_n$, it follows that

$$(\mathbb{F}_{q_1} \otimes_{\kappa_L} \ldots \otimes_{\kappa_L} \mathbb{F}_{q_n}) [\![X'_1,\ldots,X'_n]\!] \simeq \varprojlim_{m} E'^+_{\Delta_n,\circ}/(\widetilde{X}'_1,\ldots,\widetilde{X}'_n)^m E'^+_{\Delta_n,\circ}.$$

Since $X_i \in (X'_i)^{s_i}(E'^+_i)^\times$ for some $s_i \in \mathbb{N}$, it follows that we also have an isomorphism

$$(\mathbb{F}_{q_1} \otimes_{\kappa_L} \ldots \otimes_{\kappa_L} \mathbb{F}_{q_n}) [\![X'_1,\ldots,X'_n]\!] \simeq \varprojlim_{m} E'^+_{\Delta_n,\circ}/(\widetilde{X}_1,\ldots,\widetilde{X}_n)^m E'^+_{\Delta_n,\circ}.$$

On the other hand, $E'^+_{\Delta_n,\circ}$ is a finite free $E^+_{\Delta_n,\circ}$-module and thus

$$\begin{aligned}
E'^+_{\Delta_n} &= E'^+_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} E^+_{\Delta_n} \\
&\simeq E'^+_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} \varprojlim_{m} E^+_{\Delta_n,\circ}/(\widetilde{X}_1,\ldots,\widetilde{X}_n)^m E^+_{\Delta_n,\circ} \\
&\simeq \varprojlim_{m} E'^+_{\Delta_n,\circ} \otimes_{E^+_{\Delta_n,\circ}} \left(E^+_{\Delta_n,\circ}/(\widetilde{X}_1,\ldots,\widetilde{X}_n)^m E^+_{\Delta_n,\circ} \right) \\
&\simeq \varprojlim_{m} E'^+_{\Delta_n,\circ}/(\widetilde{X}_1,\ldots,\widetilde{X}_n)^m E'^+_{\Delta_n,\circ},
\end{aligned}$$

where in the second line we used the result of part (i).

(iii) This follows directly from Lemma 3.15 (i) applied for $k = \kappa_L$ and $k_i = \mathbb{F}_{q_i}$ when $i \in \Delta_n$. $\square$

Analogous to being able to embed E_L^{sep} into $\mathbb{C}_p^\flat$, in the multivariable case we will show that we can also embed $E_{\Delta_n}^{\text{sep}}$ into $\mathbb{C}_{p,\Delta_n}^\flat$. Consider the map

$$\begin{aligned}\Psi : E_{\Delta_n}^{\text{sep}} = E_{\Delta_n,\circ}^{\text{sep}} \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n} &\to \mathbb{C}_{p,\Delta_n}^\flat \\ (e_1 \otimes \ldots \otimes e_n) \otimes f &\mapsto y = (y_0, y_1, \ldots)\end{aligned}$$

where $e_i \in E_i^{\text{sep}}$ and $f \in E_{\Delta_n}$ which can be written as

$$f = \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} c_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}$$

for $c_{i_1\ldots i_n} \in \kappa_L$, then

$$y_j \coloneqq \sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{Z}^n \\ i_1+\ldots+i_n\le j}} c_{i_1\ldots i_n} (e_1 X_1^{i_1} \otimes \ldots \otimes e_n X_n^{i_n}) \in \mathbb{C}_{p,\Delta_n,\circ}^\flat.$$

It is clear that the elements y_j indeed form a Cauchy sequence of elements in $\mathbb{C}_{p,\Delta_n,\circ}^\flat$. Before proving that Ψ is well defined, we also make the following definition.

Definition 3.17. *For* $f(X_1,\ldots,X_n) = \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} c_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n} \in E_{\Delta_n}$ *we let*

$$\deg(f) \coloneqq \min\{i_1 + \ldots + i_n : c_{i_1\ldots i_n} \neq 0\}.$$

Lemma 3.18. *The map Ψ is a well-defined κ_L-algebra homomorphism.*

Proof. Using Lemma 3.7, we know that a sequence $(y_j)_{j\ge1}$ of elements in $\mathbb{C}_{p,\Delta_n,\circ}^\flat$ is Cauchy if and only if $(y_j(X_1^m \otimes \ldots \otimes X_n^m))_{j\ge1}$ is Cauchy for a given $m \in \mathbb{N}$. Therefore, to prove that Ψ is well-defined it suffices to show that Ψ is well-defined on elements of $E_{\Delta_n}'^{+}$.

Comparing Ψ with the isomorphism

$$E_{\Delta_n}'^{+} \simeq \varprojlim_m E_{\Delta_n,\circ}'^{+}/(\widetilde{X}_1,\ldots,\widetilde{X}_n)^m E_{\Delta_n,\circ}'^{+}$$

from the proof of Corollary 3.16 (ii), we conclude that Ψ is well-defined.

It is also clear that Ψ is κ_L-linear and preserves the multiplicative identities. It remains to show that it is multiplicative, for which it is sufficient to check the pure tensors.

We let $(e_1 \otimes \ldots \otimes e_n) \otimes f$ and $(e_1' \otimes \ldots \otimes e_n') \otimes f'$ in $E_{\Delta_n}^{\text{sep}}$ where $e_i, e_i' \in E_i^{\text{sep}}$, while $f, f' \in E_{\Delta_n}$ are written as

$$\begin{aligned} f &= \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} c_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}, \\ f' &= \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} c'_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}\end{aligned}$$

for $c_{i_1\dots i_n}, c'_{i_1\dots i_n} \in \kappa_L$. Write

$$ff' = \sum_{(i_1,\dots,i_n)\in\mathbb{Z}^n} c''_{i_1\dots i_n} X_1^{i_1}\dots X_n^{i_n}$$

with $c''_{i_1\dots i_n} \in \kappa_L$. Since $\deg(f)$ and $\deg(f')$ are finite (possibly negative), it follows that the deg value of

$$\sum_{\substack{(i_1,\dots,i_n)\in\mathbb{Z}^n\\ i_1+\dots+i_n\le j}} c''_{i_1\dots i_n} X_1^{i_1}\dots X_n^{i_n} - \left(\sum_{\substack{(i_1,\dots,i_n)\in\mathbb{Z}^n\\ i_1+\dots+i_n\le j}} c_{i_1\dots i_n} X_1^{i_1}\dots X_n^{i_n}\right)\left(\sum_{\substack{(i_1,\dots,i_n)\in\mathbb{Z}^n\\ i_1+\dots+i_n\le j}} c'_{i_1\dots i_n} X_1^{i_1}\dots X_n^{i_n}\right)$$

goes to ∞ as $j\to\infty$. Therefore

$$\begin{aligned}
&\Psi\big(\,((e_1\otimes\dots\otimes e_n)\otimes f)\cdot((e'_1\otimes\dots\otimes e'_n)\otimes f')\,\big) = \Psi((e_1e'_1\otimes\dots\otimes e_ne'_n)\otimes ff')\\
&= \left(\sum_{\substack{(i_1,\dots,i_n)\in\mathbb{Z}^n\\ i_1+\dots+i_n\le j}} c''_{i_1\dots i_n}(e_1e'_1X_1^{i_1}\otimes\dots\otimes e_ne'_nX_n^{i_n})\right)_{j\ge 0}\\
&= \left(\sum_{\substack{(i_1,\dots,i_n)\in\mathbb{Z}^n\\ i_1+\dots+i_n\le j}} c_{i_1\dots i_n}(e_1X_1^{i_1}\otimes\dots\otimes e_nX_n^{i_n})\right)_{j\ge 0}\left(\sum_{\substack{(i_1,\dots,i_n)\in\mathbb{Z}^n\\ i_1+\dots+i_n\le j}} c'_{i_1\dots i_n}(e'_1X_1^{i_1}\otimes\dots\otimes e'_nX_n^{i_n})\right)_{j\ge 0}\\
&= \Psi\big((e_1\otimes\dots\otimes e_n)\otimes f\big)\Psi\big((e'_1\otimes\dots\otimes e'_n)\otimes f'\big),
\end{aligned}$$

as desired. □

We will show that Ψ is an embedding. The key will be to show that the set of Cauchy sequences with elements in $E'^+_{\Delta_n,\circ}$ with respect to $|\cdot|_{\text{prod}}$ is isomorphic to $E'^+_{\Delta_n}$. This is another way of regarding $E'^+_{\Delta_n}$ as a completed tensor product. For that we require some more computations related to $|\cdot|_{\text{prod}}$.

Lemma 3.19. *Let*

$$f(X'_1,\dots,X'_n) = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} c_{i_1\dots i_n}\,(X'_1)^{i_1}\otimes\dots\otimes(X'_n)^{i_n} \in \bigotimes_{i\in\Delta_n,\kappa_L}\mathbb{F}_{q_i}[X'_i]$$

for $c_{i_1\dots i_n}\in\bigotimes_{i\in\Delta_n,\kappa_L}\mathbb{F}_{q_i}$. Then

$$|f(X'_1,\dots,X'_n)|_{\text{prod}} = \max\{|X'_1|_\flat^{i_1}\dots|X'_n|_\flat^{i_n} : c_{i_1\dots i_n}\ne 0\}.$$

Proof. It is clear that

$$|f(X'_1,\dots,X'_n)|_{\text{prod}} \le \max\{|X'_1|_\flat^{i_1}\dots|X'_n|_\flat^{i_n} : c_{i_1\dots i_n}\ne 0\}.$$

Choose a tuple $(j_1, \ldots, j_n) \in \mathbb{N}^n$ for which

$$|X_1'|_\flat^{j_1} \ldots |X_n'|_\flat^{j_n} = \max\{|X_1'|_\flat^{i_1} \ldots |X_n'|_\flat^{i_n} : c_{i_1 \ldots i_n} \neq 0\}.$$

Then for every tuple $(i_1, \ldots, i_n) \in \mathbb{N}^n - \{(j_1, \ldots, j_n)\}$ we either have that $c_{i_1 \ldots i_n} = 0$ or

$$|X_1'|_\flat^{i_1} \ldots |X_n'|_\flat^{i_n} \leq |X_1'|_\flat^{j_1} \ldots |X_n'|_\flat^{j_n}.$$

This means that $i_\ell > j_\ell$ for some $\ell \in \Delta_n$ because $|X_i'|_\flat < 1$ for $i \in \Delta_n$. Assume that

$$|f(X_1', \ldots, X_n')|_{\mathrm{prod}} < |X_1'|_\flat^{j_1} \ldots |X_n'|_\flat^{j_n},$$

then we can write

$$f(X_1', \ldots, X_n') = \sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} y_{ij} \tag{3.11}$$

for some nonzero $y_{ij} \in \mathbb{C}_p^\flat$ such that

$$\prod_{i \in \Delta_n} |y_{ij}|_\flat < |X_1'|_\flat^{j_1} \ldots |X_n'|_\flat^{j_n}$$

for all $1 \leq j \leq r$. Suppose that $m \in \mathbb{N}$ is large enough so that $(X_i')^m \, y_{ij} \in \mathcal{O}_{\mathbb{C}_p^\flat}$ for every i and j. Multiplying both sides of (3.11) by $\bigotimes_{i \in \Delta_n} (X_i')^m$, we obtain that

$$\sum_{(i_1, \ldots, i_n) \in \mathbb{N}^n} c_{i_1 \ldots i_n} \, (X_1')^{i_1+m} \otimes \ldots \otimes (X_n')^{i_n+m} = \sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} \widetilde{y}_{ij} \tag{3.12}$$

where $\widetilde{y}_{ij} \coloneqq (X_i')^m \, y_{ij}$. Then we have that

$$\prod_{i \in \Delta_n} |\widetilde{y}_{ij}|_\flat < |X_1'|_\flat^{j_1+m} \ldots |X_n'|_\flat^{j_n+m}$$

for all $1 \leq j \leq r$. Choose $\ell \in \mathbb{N}$ large enough such that

$$\prod_{i \in \Delta_n} |\widetilde{y}_{ij}|_\flat < |X_1'|_\flat^{j_1+m+\frac{1}{\ell}} \ldots |X_n'|_\flat^{j_n+m+\frac{1}{\ell}} \tag{3.13}$$

still holds for all $1 \leq j \leq r$. Let $(X_i')^{\frac{1}{\ell}}$ be an ℓ-th root of X_i', which exists since $\mathbb{C}_p^\flat$ is algebraically closed. Consider the projection map

$$\mathrm{pr} : \mathcal{O}_{\mathbb{C}_p^\flat, \Delta_n, \circ} \to \bigotimes_{i \in \Delta_n, \kappa_L} \mathcal{O}_{\mathbb{C}_p^\flat} / \, (X_i')^{j_i+m+\frac{1}{\ell}} \, \mathcal{O}_{\mathbb{C}_p^\flat}.$$

By (3.13) it follows that for every $1 \leq j \leq r$, there exists an index $i \in \Delta_n$ (depending on j) such that $\widetilde{y}_{ij} \in (X_i')^{j_i+m+\frac{1}{\ell}} \, \mathcal{O}_{\mathbb{C}_p^\flat}$. Therefore

$$\mathrm{pr}\left(\sum_{(i_1, \ldots, i_n) \in \mathbb{N}^n} c_{i_1 \ldots i_n} \, (X_1')^{i_1+m} \otimes \ldots \otimes (X_n')^{i_n+m} \right) = \mathrm{pr}\left(\sum_{j=1}^{r} \bigotimes_{i \in \Delta_n} \widetilde{y}_{ij} \right) = 0.$$

On the other hand we also have that

$$\begin{aligned}
&\mathrm{pr}\left(\sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n} c_{i_1\ldots i_n}\left(X_1'\right)^{i_1+m}\otimes\ldots\otimes\left(X_n'\right)^{i_n+m}\right)\\
&=\mathrm{pr}\left(\sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{N}^n\\ i_1\le j_1,\ldots,i_n\le j_n}} c_{i_1\ldots i_n}\left(X_1'\right)^{i_1+m}\otimes\ldots\otimes\left(X_n'\right)^{i_n+m}\right)\\
&=\mathrm{pr}\left(c_{j_1\ldots j_n}\left(X_1'\right)^{j_1+m}\otimes\ldots\otimes\left(X_n'\right)^{j_n+m}\right),
\end{aligned}$$

where the last equality holds since $c_{i_1\ldots i_n}=0$ for $(i_1,\ldots,i_n)\in\mathbb{N}^n-\{(j_1,\ldots,j_n)\}$ with $i_1\le j_1,\ldots,i_n\le j_n$ by our maximality assumption about the tuple $(j_1,\ldots,j_n)$. Regarding

$$\mathrm{pr}\left(c_{j_1\ldots j_n}\left(X_1'\right)^{j_1+m}\otimes\ldots\otimes\left(X_n'\right)^{j_n+m}\right)$$

as an element of $\bigotimes\limits_{i\in\Delta_n,\kappa_L}\mathbb{F}_{q_i}\left[\left(X_i'\right)^{\frac{1}{\ell}}\right]/\left(X_i'\right)^{j_i+m+\frac{1}{\ell}}\mathbb{F}_{q_i}\left[\left(X_i'\right)^{\frac{1}{\ell}}\right]$, under the identification

$$\bigotimes_{i\in\Delta_n,\kappa_L}\mathbb{F}_{q_i}\left[\left(X_i'\right)^{\frac{1}{\ell}}\right]\simeq\left(\bigotimes_{i\in\Delta_n,\kappa_L}\mathbb{F}_{q_i}\right)\left[\left(X_1'\right)^{\frac{1}{\ell}},\ldots,\left(X_n'\right)^{\frac{1}{\ell}}\right]$$

the element $c_{j_1\ldots j_n}\left(X_1'\right)^{j_1+m}\otimes\ldots\otimes\left(X_n'\right)^{j_n+m}$ is identified with $c_{j_1\ldots j_n}\left(X_1'\right)^{j_1+m}\ldots\left(X_n'\right)^{j_n+m}$. Since

$$\mathrm{pr}\left(c_{j_1\ldots j_n}\left(X_1'\right)^{j_1+m}\otimes\ldots\otimes\left(X_n'\right)^{j_n+m}\right)=0$$

it is then clear by the above identification that $c_{j_1\ldots j_n}\left(X_1'\right)^{j_1+m}\ldots\left(X_n'\right)^{j_n+m}=0$ in $\left(\bigotimes_{i\in\Delta_n,\kappa_L}\mathbb{F}_{q_i}\right)\left[\left(X_1'\right)^{\frac{1}{\ell}},\ldots,\left(X_n'\right)^{\frac{1}{\ell}}\right]$. The latter holds only if $c_{j_1\ldots j_n}=0$ and therefore we get the desired contradiction. □

Lemma 3.20. *Let $f\in E'^{+}_{\Delta_n,\circ}$ and*

$$\begin{aligned}
\psi:\mathbb{F}_{q_1}[\![X_1']\!]\otimes_{\kappa_L}\ldots\otimes_{\kappa_L}\mathbb{F}_{q_n}[\![X_n']\!]&\longrightarrow\left(\mathbb{F}_{q_1}\otimes_{\kappa_L}\ldots\otimes_{\kappa_L}\mathbb{F}_{q_n}\right)[\![X_1',\ldots,X_n']\!]\\
\left(\sum_{j_1=0}^{\infty}c_{1j_1}(X_1')^{j_1}\right)\otimes\ldots\otimes\left(\sum_{j_n=0}^{\infty}c_{nj_n}(X_n')^{j_n}\right)&\longmapsto\sum_{(j_1,\ldots,j_n)\in\mathbb{N}^n}\left(c_{1j_1}\otimes\ldots\otimes c_{nj_n}\right)(X_1')^{j_1}\ldots(X_n')^{j_n}
\end{aligned}$$

be the natural embedding. Then if we write

$$\psi(f)=\sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n}c_{i_1\ldots i_n}\left(X_1'\right)^{i_1}\ldots\left(X_n'\right)^{i_n}$$

with $c_{i_1\ldots i_n}\in\bigotimes\limits_{i\in\Delta_n,\kappa_L}\mathbb{F}_{q_i}$, we have that

$$|f|_{\mathrm{prod}}=\max\{|X_1'|_\flat^{i_1}\ldots|X_n'|_\flat^{i_n}:c_{i_1\ldots i_n}\neq0\}.$$

Proof. First, we choose a tuple $(j_1, \dots, j_n) \in \mathbb{N}^n$ for which

$$|X_1'|_\flat^{j_1} \dots |X_n'|_\flat^{j_n} = \max\{|X_1'|_\flat^{i_1} \dots |X_n'|_\flat^{i_n} : c_{i_1 \dots i_n} \neq 0\}.$$

Let $m \in \mathbb{N}$ be large enough so that

$$|X_i'|_\flat^m < |X_1'|_\flat^{j_1} \dots |X_n'|_\flat^{j_n} \tag{3.14}$$

for all $i \in \Delta_n$. Then $m > j_i$ for all $i \in \Delta_n$. We can write

$$f = \sum_{\substack{(i_1,\dots i_n) \in \mathbb{N}^n \\ i_1,\dots,i_n \leq m-1}} c_{i_1\dots i_n} \, (X_1')^{i_1} \otimes \dots \otimes (X_n')^{i_n} + \big((X_1')^m \otimes 1 \otimes \dots \otimes 1\big) \, g_1 + \dots + \big(1 \otimes \dots \otimes 1 \otimes (X_n')^m\big) \, g_n$$

for some $g_i \in E'^+_{\Delta_n,\circ}$. By Lemma 3.19 we know that

$$|\sum_{\substack{(i_1,\dots i_n) \in \mathbb{N}^n \\ i_1,\dots,i_n \leq m-1}} c_{i_1\dots i_n} \, (X_1')^{i_1} \otimes \dots \otimes (X_n')^{i_n} \, |_{\mathrm{prod}} = |X_1'|_\flat^{j_1} \dots |X_n'|_\flat^{j_n}.$$

It is also clear that $|g_i|_{\mathrm{prod}} \leq 1$ because $g_i \in E'^+_{\Delta_n,\circ}$. Therefore

$$|\,\big((X_1')^m \otimes 1 \otimes \dots \otimes 1\big) \, g_1 + \dots + \big(1 \otimes \dots \otimes 1 \otimes (X_n')^m\big) \, g_n|_{\mathrm{prod}} < |X_1'|_\flat^{j_1} \dots |X_n'|_\flat^{j_n}$$

by (3.14) and thus

$$|f|_{\mathrm{prod}} = |X_1'|_\flat^{j_1} \dots |X_n'|_\flat^{j_n}$$

by the nonarchimedean property. □

We are finally ready to show that Ψ is an embedding.

Proposition 3.21. *The map $\Psi : E^{\mathrm{sep}}_{\Delta_n} \to \mathbb{C}^\flat_{p,\Delta_n}$ is an embedding.*

Proof. The restriction of Ψ to $E'^+_{\Delta_n}$ is the natural map from $E'^+_{\Delta_n}$ into the set of equivalence classes of Cauchy sequences with elements in $E'^+_{\Delta_n,\circ}$ with respect to $|\cdot|_{\mathrm{prod}}$. Lemma 3.20 shows that if we write $E_i' = \mathbb{F}_{q_i}(\!(X_i')\!)$ for $i \in \Delta_n$, then the aforementioned set of equivalence classes of Cauchy sequences is isomorphic to $(\mathbb{F}_{q_1} \otimes_{\kappa_L} \dots \otimes_{\kappa_L} \mathbb{F}_{q_n}) [\![X_1', \dots, X_n']\!]$. The restriction of Ψ to $E'^+_{\Delta_n}$ becomes the map from $E'^+_{\Delta_n}$ into $(\mathbb{F}_{q_1} \otimes_{\kappa_L} \dots \otimes_{\kappa_L} \mathbb{F}_{q_n}) [\![X_1', \dots, X_n']\!]$ that we showed in Corollary 3.16 (ii) to be an isomorphism. Therefore, the restriction of Ψ to $E'^+_{\Delta_n}$ is an embedding.

For the restriction of Ψ on E'_{Δ_n}, one notes that if $z \in E'_{\Delta_n}$, there exists a large enough m such that $X^m_{\Delta_n} z \in E'^+_{\Delta_n}$ by Lemma 3.13 (ii). Because $\Psi(X_{\Delta_n})$ is a unit in $\mathbb{C}^\flat_{p,\Delta_n}$, we have that $\Psi(z) = 0$ if and only if $\Psi(X^m_{\Delta_n} z) = 0$. By what we showed above, that means that $X^m_{\Delta_n} z = 0$ or that $z = 0$. Hence Ψ is injective on E'_{Δ_n} as well. Our claim follows using a colimit argument. □

From now on we identify $E^{\mathrm{sep}}_{\Delta_n}$ with its image in $\mathbb{C}^\flat_{p,\Delta_n}$. We have the following analog of Lemma 1.39 in the multivariable case.

Lemma 3.22. *The action of the group $G_{\Delta_n,L}$ on $\mathbb{C}^\flat_{p,\Delta_n}$ preserves $E^{\text{sep}}_{\Delta_n}$ and $E^{\text{sep}+}_{\Delta_n}$. If $f = f(X_1, \ldots, X_n) \in E_{\Delta_n}$, $e_i \in E^{\text{sep}}_i$ and $\sigma_i \in G_{i,L}$ for $i \in \Delta_n$, then*

$$\left(\prod_{i\in\Delta_n} \sigma_i\right)((e_1\otimes\ldots\otimes e_n)\otimes f) = (\sigma_1(e_1)\otimes\ldots\otimes\sigma_n(e_n))\otimes f\left(\overline{[\chi_L(\overline{\sigma_1})]}_{\phi_1}(X_1), \ldots, \overline{[\chi_L(\overline{\sigma_n})]}_{\phi_n}(X_n)\right)$$

where $\overline{\sigma_i} = \sigma_i \bmod H_{i,L}$.

Proof. By Lemma 1.39 we know that the right hand side of the claimed formula lies in $E^{\text{sep}}_{\Delta_n}$ and we will show that the two sides are equal using the embedding Ψ. Suppose that $f \in E_{\Delta_n}$ is written as

$$f = \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} c_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}$$

for some $c_{i_1\ldots i_n} \in \kappa_L$. Suppose that we also write

$$f\left(\overline{[\chi_L(\overline{\sigma_1})]}_{\phi_1}(X_1), \ldots, \overline{[\chi_L(\overline{\sigma_n})]}_{\phi_n}(X_n)\right) = \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} d_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}.$$

for some $d_{i_1\ldots i_n} \in \kappa_L$. Since

$$\sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} c_{i_1\ldots i_n}\overline{[\chi_L(\overline{\sigma_1})]}_{\phi_1}(X_1)^{i_1} \ldots \overline{[\chi_L(\overline{\sigma_n})]}_{\phi_n}(X_n)^{i_n} = \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} d_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}$$

and $\overline{[\chi_L(\overline{\sigma_1})]}_{\phi_1}(X_1)^{i_1} \ldots \overline{[\chi_L(\overline{\sigma_n})]}_{\phi_n}(X_n)^{i_n} \in X_1^{i_1} \ldots X_n^{i_n} E^+_{\Delta_n}$, the value

$$\deg\left(\sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{Z}^n\\ i_1+\ldots+i_n\le j}} c_{i_1\ldots i_n}\overline{[\chi_L(\overline{\sigma_1})]}_{\phi_1}(X_1)^{i_1} \ldots \overline{[\chi_L(\overline{\sigma_n})]}_{\phi_n}(X_n)^{i_n} - \sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{Z}^n\\ i_1+\ldots+i_n\le j}} d_{i_1\ldots i_n} X_1^{i_1} \ldots X_n^{i_n}\right)$$

goes to ∞ as $j \to \infty$. Therefore we have the equalities

$$\left(\prod_{i\in\Delta_n} \sigma_i\right)\left(\Psi\left((e_1\otimes\ldots\otimes e_n)\otimes f\right)\right) = \left(\prod_{i\in\Delta_n} \sigma_i\right)\left(\sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{Z}^n\\ i_1+\ldots+i_n\le j}} c_{i_1\ldots i_n}(e_1X_1^{i_1}\otimes\ldots\otimes e_nX_n^{i_n})\right)_{j\ge0}$$

$$= \left(\sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{Z}^n\\ i_1+\ldots+i_n\le j}} c_{i_1\ldots i_n}\left(\sigma_1(e_1)\cdot\overline{[\chi_L(\overline{\sigma_1})]}_{\phi_1}(X_1)^{i_1}\otimes\ldots\otimes\sigma_n(e_n)\cdot\overline{[\chi_L(\overline{\sigma_n})]}_{\phi_n}(X_n)^{i_n}\right)\right)_{j\ge0}$$

$$= \left(\sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{Z}^n\\ i_1+\ldots+i_n\le j}} d_{i_1\ldots i_n}\left(\sigma_1(e_1)X_1^{i_1}\otimes\ldots\otimes\sigma_n(e_n)X_n^{i_n}\right)\right)_{j\ge0}$$

$$= \Psi\left((\sigma_1(e_1)\otimes\ldots\otimes\sigma_n(e_n))\otimes f\left(\overline{[\chi_L(\overline{\sigma_1})]}_{\phi_1}(X_1), \ldots, \overline{[\chi_L(\overline{\sigma_n})]}_{\phi_n}(X_n)\right)\right)$$

where the second equality is Lemma 1.38 (ii). If $f \in E^+_{\Delta_n}$, it is clear that

$$f\left(\overline{[\chi_L(\overline{\sigma_1})]}_{\phi_1}(X_1), \dots, \overline{[\chi_L(\overline{\sigma_n})]}_{\phi_n}(X_n)\right) \in E^+_{\Delta_n}.$$

By Lemma 1.39 we know that $\sigma(e_i) \in E_i^{\mathrm{sep}\,+}$ when $e_i \in E_i^{\mathrm{sep}\,+}$, therefore the action of $G_{\Delta_n,L}$ preserves $E_{\Delta_n}^{\mathrm{sep}\,+}$ as well. □

In particular, Lemma 3.22 implies that the action of the group $G_{\Delta_n,L}$ on $\mathbb{C}^\flat_{p,\Delta_n}$ preserves E_{Δ_n} and that

$$\left(\prod_{i\in\Delta_n} \sigma_i\right) \circ f(X_1, \dots, X_n) = f\left(\overline{[\chi_L(\overline{\sigma_1})]}_{\phi_1}(X_1), \dots, \overline{[\chi_L(\overline{\sigma_n})]}_{\phi_n}(X_n)\right) \tag{3.15}$$

for $f(X_1, \dots, X_n) \in E_{\Delta_n}$. Before we end the section, we make an observation about the topology on E_{Δ_n} induced by the topology on $\mathbb{C}^\flat_{p,\Delta_n}$.

Lemma 3.23. *The induced topology on E_{Δ_n} by the norm $|\cdot|_{\mathrm{prod}}$ on $\mathbb{C}^\flat_{p,\Delta_n}$ coincides with the weak topology on E_{Δ_n}.*

Proof. Specializing the results of Lemma 3.20 for $E^+_{\Delta_n}$ by taking the collection of trivial extensions, we get that the norm of

$$f(X_1, \dots, X_n) = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} c_{i_1\dots i_n} X_1^{i_1} \dots X_n^{i_n} \in E^+_{\Delta_n}$$

equals

$$|f(X_1, \dots, X_n)|_{\mathrm{prod}} = |\omega|_\flat^{\deg(f)} \tag{3.16}$$

because $|X_i|_\flat = |\omega|_\flat$ for all $i \in \Delta_n$. Using Lemma 3.7, we get that (3.16) holds for all $f \in E_{\Delta_n}$.

Fix an $\varepsilon \in \mathbb{R}_{>0}$ and let ℓ be the smallest integer in $\mathbb{Z}$ such that $|\omega|_\flat^\ell \le \varepsilon$. By (3.16) it is clear that for every $g \in E_{\Delta_n,\ell}$ we have that $|g|_{\mathrm{prod}} \le \varepsilon$. Conversely, if $g \in E_{\Delta_n}$ is such that $|g|_{\mathrm{prod}} \le \varepsilon$ then because $|\cdot|_{\mathrm{prod}}$ takes values in $|\omega|_\flat^{\mathbb{Z}}$, we have that $|g|_{\mathrm{prod}} \le |\omega|_\flat^\ell$ and thus $g \in E_{\Delta_n,\ell}$ by (3.16). The conclusion follows. □

3.3 The ring $W(\mathbb{C}^\flat_{p,\Delta_n})_L$ and its weak topology

In the one variable case, we were also able to embed all the relevant rings of characteristic zero in the ring of ramified Witt vectors $W(\mathbb{C}^\flat_p)_L$. In the multivariable case, we consider $W(\mathbb{C}^\flat_{p,\Delta_n})_L \left(= \prod_{m\ge0} \mathbb{C}^\flat_{p,\Delta_n} \text{ as sets}\right)$ which we equip with the product topology of the topologies induced by $|\cdot|_{\mathrm{prod}}$ on each component $\mathbb{C}^\flat_{p,\Delta_n}$ and call this the *weak topology* on $W(\mathbb{C}^\flat_{p,\Delta_n})_L$. Since $G_{\Delta_n,L}$ acts on $\mathbb{C}^\flat_{p,\Delta_n}$, by functoriality of the Witt vectors construction $G_{\Delta_n,L}$ acts on $W(\mathbb{C}^\flat_{p,\Delta_n})_L$ as well.

Lemma 3.24. *The group $G_{\Delta_n,L}$ acts continuously on $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$.*

Proof. Let $\sigma \in G_{\Delta_n,L}$ and $(a_0, a_1, \ldots) \in W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ for some $a_j \in \mathbb{C}_{p,\Delta_n}^\flat$. Suppose that $V_0, \ldots, V_m$ are open neighbourhoods of $\sigma(a_0), \ldots, \sigma(a_m)$, respectively. By continuity of the action of $G_{\Delta_n,L}$ on $\mathbb{C}_{p,\Delta_n}^\flat$, we know that there exist open neighbourhoods $U_0, \ldots, U_m$ of $a_0, \ldots, a_m$, respectively, and open subgroups $H_0, \ldots, H_m$ of $G_{\Delta_n,L}$ such that

$$\sigma H_j(U_j) \subseteq V_j$$

for all $0 \leq j \leq m$. Letting $H \coloneqq \bigcap_{j=0}^{m} H_j$, we obtain that

$$\sigma H\left(U_0 \times \ldots \times U_m \times \mathbb{C}_{p,\Delta_n}^\flat \times \mathbb{C}_{p,\Delta_n}^\flat \times \ldots\right) \subseteq V_0 \times \ldots \times V_m \times \mathbb{C}_{p,\Delta_n}^\flat \times \mathbb{C}_{p,\Delta_n}^\flat \times \ldots,$$

thus proving the continuity. □

For the weak topology on $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$, the sets

$$\mathfrak{U}_{\varepsilon,m,\mathbf{a}} = \{(b_0, b_1, \ldots) \in W(\mathbb{C}_{p,\Delta_n}^\flat)_L : |b_j - a_j|_{\mathrm{prod}} < \varepsilon \text{ for } 0 \leq j \leq m\}$$

form a fundamental system of open neighbourhoods of $\mathbf{a} = (a_0, a_1, \ldots) \in W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ where $\varepsilon \in \mathbb{R}_{>0}, m \in \mathbb{N}$.

Lemma 3.25. $(W(\mathbb{C}_{p,\Delta_n}^\flat)_L, \boxplus, \boxdot)$ *is a topological ring for the weak topology.*

Proof. We first prove that addition is continuous on $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$. Let $\mathbf{a} = (a_0, a_1, \ldots)$ and $\mathbf{b} = (b_0, b_1, \ldots)$ in $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ and suppose that

$$\mathbf{a} \boxplus \mathbf{b} = \mathbf{z} = (z_0, z_1, \ldots).$$

Consider a basic open neighbourhood $\mathfrak{U}_{\varepsilon,m,\mathbf{z}}$ for some $m \geq 0$ and $0 < \varepsilon \leq 1$. Let

$$\delta' \coloneqq \max\{|a_j|_{\mathrm{prod}}, |b_j|_{\mathrm{prod}} : 0 \leq j \leq m\}$$

and $\delta \coloneqq \max\{1, \delta'\}$. We will show that

$$\mathfrak{U}_{\varepsilon\delta^{-(q^m-1)},m,\mathbf{a}} \boxplus \mathfrak{U}_{\varepsilon\delta^{-(q^m-1)},m,\mathbf{b}} \subseteq \mathfrak{U}_{\varepsilon,m,\mathbf{z}} \tag{3.17}$$

which would prove the continuity of addition at $(\mathbf{a}, \mathbf{b})$. Let $\mathbf{x} \in \mathfrak{U}_{\varepsilon\delta^{-(q^m-1)},m,\mathbf{a}}$ and $\mathbf{y} \in \mathfrak{U}_{\varepsilon\delta^{-(q^m-1)},m,\mathbf{b}}$, then we can write

$$\mathbf{x} = (a_0 + c_0, \ldots, a_m + c_m, x_{m+1}, x_{m+2}, \ldots),$$
$$\mathbf{y} = (b_0 + d_0, \ldots, b_m + d_m, y_{m+1}, y_{m+2}, \ldots)$$

for some $c_0, \ldots, c_m, d_0, \ldots, d_m, x_{m+1}, x_{m+2}, \ldots, y_{m+1}, y_{m+2}, \ldots \in \mathbb{C}_{p,\Delta_n}^\flat$ such that

$$|c_j|_{\mathrm{prod}}, |d_j|_{\mathrm{prod}} < \varepsilon\delta^{-(q^m-1)}$$

for $0 \leq j \leq m$. Suppose that

$$\mathbf{x} \boxplus \mathbf{y} = \mathbf{w} = (w_0, w_1, \ldots).$$

We need to show that $|w_j - z_j|_{\text{prod}} < \varepsilon$ for $0 \leq j \leq m$. Recall from the addition formulas of ramified Witt vectors that

$$z_j = S_j(a_0, \ldots, a_j, b_0, \ldots, b_j)$$

and

$$w_j = S_j(a_0 + c_0, \ldots, a_j + c_j, b_0 + d_0, \ldots, b_j + d_j)$$

for $0 \leq j \leq m$. Fix a $0 \leq j \leq m$. For $r_0, \ldots, r_j, s_0, \ldots, s_j \in \mathbb{N}$ such that $r_0 + \ldots, r_j + s_0 + \ldots + s_j \leq q^j$, we have that

$$\left|(a_0 + c_0)^{r_0} \ldots (a_j + c_j)^{r_j} (b_0 + d_0)^{s_0} \ldots (b_j + d_j)^{s_j} - a_0^{r_0} \ldots a_j^{r_j} b_0^{s_0} \ldots b_j^{s_j}\right|_{\text{prod}} < \varepsilon. \tag{3.18}$$

Indeed, fix a term in the expansion of the above expression. It contains at least one of the $c_0, \ldots, c_j, d_0, \ldots, d_j$ as a factor, which contributes an absolute value less than $\varepsilon\delta^{-(q^m-1)}$. The other factors in the term are either from $a_0, \ldots, a_j, b_0, \ldots, b_j$ whose absolute value is at most δ or from $c_0, \ldots, c_j, d_0, \ldots, d_j$, whose absolute value is at most 1 because $\varepsilon \leq 1$ and $\delta \geq 1$. In such a term, there are at most $q^j - 1$ such other factors, therefore the absolute value of the term is less than $\varepsilon \frac{\delta^{q^j-1}}{\delta^{q^m-1}} \leq \varepsilon$.

By Lemma 1.4 (i), the degree of S_j is at most q^j. Therefore we have that

$$|S_j(a_0 + c_0, \ldots, a_j + c_j, b_0 + d_0, \ldots, b_j + d_j) - S_j(a_0, \ldots, a_j, b_0, \ldots, b_j)|_{\text{prod}} < \varepsilon,$$

by (3.18) and the nonarchimedean property of $|\cdot|_{\text{prod}}$. Therefore (3.17) holds, as desired.

To show that multiplication is continuous on $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$, one applies an analogous argument to the one used for addition aided by Lemma 1.4 (iii) this time. $\square$

Remark 3.26. (i) It follows that the theory of Section 1.4 applies for the ring $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ and we call the linear topology of a finitely generated $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$-module the *weak topology*.

(ii) One can show that a formula analogous to (3.17) holds for additive inverses as well. We let $\mathbf{a} = (a_0, a_1, \ldots) \in W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ and suppose that

$$\boxminus \mathbf{a} = \mathbf{z} = (z_0, z_1, \ldots).$$

Consider a basic open neighbourhood $\mathfrak{U}_{\varepsilon,m,\mathbf{z}}$ for some $m \geq 0$ and $0 < \varepsilon \leq 1$ and let

$$\delta' := \max\{|a_j|_{\text{prod}} : 0 \leq j \leq m\}$$

and $\delta := \max\{1, \delta'\}$. Then we have that

$$\boxminus \mathfrak{U}_{\varepsilon\delta^{-(q^m-1)},m,\mathbf{a}} \subseteq \mathfrak{U}_{\varepsilon,m,\mathbf{z}}. \tag{3.19}$$

Indeed, if $\mathbf{x} \in \mathfrak{U}_{\varepsilon\delta^{-(q^m-1)},m,\mathbf{a}}$, we can write

$$\mathbf{x} = (a_0 + c_0, \ldots, a_m + c_m, x_{m+1}, x_{m+2}, \ldots),$$

for some $c_0, \ldots, c_m, x_{m+1}, x_{m+2}, \ldots \in \mathbb{C}^\flat_{p,\Delta_n}$ such that

$$|c_j|_{\mathrm{prod}} < \varepsilon\delta^{-(q^m-1)}$$

for $0 \leq j \leq m$. Suppose that

$$\boxminus \mathbf{x} = \mathbf{w} = (w_0, w_1, \ldots).$$

We need to show that $|w_j - z_j|_{\mathrm{prod}} < \varepsilon$ for $0 \leq j \leq m$. Recall from the additive inverse formulas of ramified Witt vectors that

$$z_j = I_j(a_0, \ldots, a_j)$$

and

$$w_j = I_j(a_0 + c_0, \ldots, a_j + c_j)$$

for $0 \leq j \leq m$. Using (3.18) for the case when $s_0 = \ldots = s_j = 0$ and the fact that the degree of I_j is at most q^j by Lemma 1.4 (ii), we have that

$$|I_j(a_0 + c_0, \ldots, a_j + c_j) - I_j(a_0, \ldots, a_j)|_{\mathrm{prod}} < \varepsilon,$$

by the nonarchimedean property of $|\cdot|_{\mathrm{prod}}$. Therefore (3.19) holds, as desired.

(iii) Specializing the formulae (3.17) and (3.19) for $\mathbf{a} = \mathbf{b} = \mathbf{0}$ we obtain that for the weak topology on $W(\mathbb{C}^\flat_{p,\Delta_n})_L$ the sets

$$(\mathfrak{U}_{\varepsilon,m,\mathbf{0}})_{0<\varepsilon\leq 1, m\in\mathbb{N}_{\geq 0}}$$

form a fundamental system of open neighbourhoods of zero that are closed under addition and additive inverses.

(iv) On $W(E_{\Delta_n})_L$ consider the subspace topology of the weak topology on $W(\mathbb{C}^\flat_{p,\Delta_n})_L$. We call this topology on $W(E_{\Delta_n})_L$ the weak topology as well. This is the product topology of the topologies on E_{Δ_n} induced by $|\cdot|_{\mathrm{prod}}$. By Lemma 3.23 this is also the product topology of the weak topologies on each factor E_{Δ_n}. By Lemma 3.25 $(W(E_{\Delta_n})_L, \boxplus, \boxdot)$ is a topological ring for this topology and the sets

$$\mathcal{U}_{\ell,m,\mathbf{a}} = \{(b_0, b_1, \ldots) \in W(E_{\Delta_n})_L : b_j \in a_j + E_{\Delta_n,\ell} \text{ for } 0 \leq j \leq m\}$$

form a fundamental system of open neighbourhoods of $\mathbf{a} = (a_0, a_1, \ldots) \in W(E_{\Delta_n})_L$ where $\ell \in \mathbb{Z}, m \in \mathbb{N}$. Let $\varepsilon > 0$ and ℓ be the smallest integer such that $|\omega|_\flat^\ell \leq \varepsilon$. Since $|\cdot|_{\mathrm{prod}}$ takes its values in $|\omega|_\flat^{\mathbb{Z}}$, we have that

$$\mathfrak{U}_{\varepsilon,m,\mathbf{a}} \cap W(E_{\Delta_n})_L = \mathcal{U}_{\ell,m,\mathbf{a}}.$$

For $\mathbf{a} = \mathbf{0}$, by part (iii) of our remark we have that

$$\mathcal{U}_{\ell,m,\mathbf{0}} \boxplus \mathcal{U}_{\ell,m,\mathbf{0}} \subseteq \mathcal{U}_{\ell,m,\mathbf{0}} \tag{3.20}$$

and

$$\boxminus \mathcal{U}_{\ell,m,\mathbf{0}} \subseteq \mathcal{U}_{\ell,m,\mathbf{0}}, \tag{3.21}$$

respectively, for $\ell, m \geq 0$.

3.4 Embedding $\mathcal{A}_{\Delta_n}$ into $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$

We now embed $\mathcal{A}_{\Delta_n}$ into $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ in a specific way. On the one hand, the ring $\mathcal{A}_{\Delta_n}$ is equipped with a $G_{\Delta_n,L}$-action through the quotient $\Gamma_{\Delta_n,L}$ as well as the Frobenius map φ. On the other hand, $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ is also equipped with a $G_{\Delta_n,L}$-action and a Frobenius map F. We want that our embedding respects these additional structures.

Let $\mathrm{j}_{\mathcal{A}_{\Delta_n}} : \mathcal{A}_{\Delta_n} \to W(E_{\Delta_n})_L$ denote the composition of

$$\mathcal{A}_{\Delta_n} \xrightarrow{s_{\mathcal{A}_{\Delta_n}}} W(\mathcal{A}_{\Delta_n})_L \xrightarrow{W(\mathrm{pr})_L} W(E_{\Delta_n})_L$$

where $s_{\mathcal{A}_{\Delta_n}}$ is the map from Proposition 1.10 applied for $B = \mathcal{A}_{\Delta_n}$ and $\psi = \varphi$, while pr is the natural projection map

$$\mathrm{pr} : \mathcal{A}_{\Delta_n} \longrightarrow \mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n} \simeq E_{\Delta_n}$$

and $W(\mathrm{pr})_L$ is the map induced by the functoriality of the ramified Witt vectors construction.

Lemma 3.27. *The map $\mathrm{j}_{\mathcal{A}_{\Delta_n}}$ is injective.*

Proof. Let $b \in \mathcal{A}_{\Delta_n}$ be such that $\mathrm{j}_{\mathcal{A}_{\Delta_n}}(b) = 0$. Then

$$0 = \Phi_0(\mathrm{j}_{\mathcal{A}_{\Delta_n}}(b)) = \Phi_0 \circ W(\mathrm{pr})_L \circ s_{\mathcal{A}_{\Delta_n}}(b) = \mathrm{pr} \circ \Phi_0 \circ s_{\mathcal{A}_{\Delta_n}}(b) = \mathrm{pr}(b)$$

where the last equality holds by the definition of $s_{\mathcal{A}_{\Delta_n}}$. Therefore $b = \pi b_1$ for some $b_1 \in \mathcal{A}_{\Delta_n}$. The map $\mathrm{j}_{\mathcal{A}_{\Delta_n}}$ is an $\mathcal{O}_L$-algebra homomorphism, therefore $0 = \mathrm{j}_{\mathcal{A}_{\Delta_n}}(\pi b_1) = \pi \mathrm{j}_{\mathcal{A}_{\Delta_n}}(b_1)$. By Lemma 1.8 (ii) and because E_{Δ_n} is reduced, it follows that $\mathrm{j}_{\mathcal{A}_{\Delta_n}}(b_1) = 0$, therefore $b_1 \in \pi\mathcal{A}_{\Delta_n}$. Continuing in the same manner we obtain that

$$b \in \bigcap_{m \geq 1} \pi^m \mathcal{A}_{\Delta_n} = 0$$

since $\mathcal{A}_{\Delta_n}$ is π-adically complete. $\square$

Proposition 3.28. *The map $\mathrm{j}_{\mathcal{A}_{\Delta_n}}$ respects the Frobenius maps and the Galois actions on both sides, in other words*

(i) $F \circ \mathrm{j}_{\mathcal{A}_{\Delta_n}}(b) = \mathrm{j}_{\mathcal{A}_{\Delta_n}} \circ \varphi(b)$,

(ii) $\sigma \circ \mathrm{j}_{\mathcal{A}_{\Delta_n}}(b) = \mathrm{j}_{\mathcal{A}_{\Delta_n}} \circ \sigma(b)$

for any $b \in \mathcal{A}_{\Delta_n}$ and $\sigma \in G_{\Delta_n,L}$.

Proof. (i) Consider the diagram

$$\begin{array}{ccccc}
\mathcal{A}_{\Delta_n} & \xrightarrow{s_{\mathcal{A}_{\Delta_n}}} & W(\mathcal{A}_{\Delta_n})_L & \xrightarrow{W(\mathrm{pr})_L} & W(E_{\Delta_n})_L \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\
\mathcal{A}_{\Delta_n} & \xrightarrow{s_{\mathcal{A}_{\Delta_n}}} & W(\mathcal{A}_{\Delta_n})_L & \xrightarrow{W(\mathrm{pr})_L} & W(E_{\Delta_n})_L,
\end{array}$$

The left square commutes by Proposition 1.10 and it is clear that the right square commutes as well. Therefore the diagram is commutative and the conclusion follows.

(ii) Write $\sigma = \prod_{i\in\Delta_n}\sigma_i$ for $\sigma_i \in G_{i,L}$. Suppose that $\chi_L(\overline{\sigma_i}) = a_i \in \mathcal{O}_L^\times$ for $i \in \Delta_n$ where $\overline{\sigma_i} = \sigma_i \bmod H_{i,L}$. Write $b = b(X_1, \ldots, X_n)$ and suppose that

$$s_{\mathcal{A}_{\Delta_n}}(b) = (u_0, u_1, \ldots)$$

for some $u_m = u_m(X_1, \ldots, X_n) \in \mathcal{A}_{\Delta_n}$. Then

$$\sigma \circ \mathrm{j}_{\mathcal{A}_{\Delta_n}}(b) = (v_0, v_1, \ldots)$$

where

$$v_m = u_m([a_1]_{\phi_1}(X_1), \ldots, [a_n]_{\phi_n}(X_n)) \bmod \pi\mathcal{A}_{\Delta_n}.$$

It suffices to show that

$$s_{\mathcal{A}_{\Delta_n}} \circ \sigma(b) = (w_0, w_1, \ldots)$$

where

$$w_m = u_m([a_1]_{\phi_1}(X_1), \ldots, [a_n]_{\phi_n}(X_n)) \in \mathcal{A}_{\Delta_n}.$$

By the defining property of the map $s_{\mathcal{A}_{\Delta_n}}$, we know that since π is not a zero divisor in $\mathcal{A}_{\Delta_n}$, the w_j are the unique elements in $\mathcal{A}_{\Delta_n}$ such that

$$\Phi_m(w_0, \ldots, w_m) = \varphi^m \circ \sigma(b)$$

for all $m \geq 0$. On the one hand, we have the equality

$$\begin{aligned}
\varphi^m \circ \sigma(b) &= \varphi^m \circ b([a_1]_{\phi_1}(X_1), \ldots, [a_n]_{\phi_n}(X_n)) \\
&= b\big([a_1]_{\phi_1}([\pi^m]_{\phi_1}(X_1)), \ldots, [a_n]_{\phi_n}([\pi^m]_{\phi_n}(X_n))\big) \\
&= b([a_1\pi^m]_{\phi_1}(X_1), \ldots, [a_n\pi^m]_{\phi_n}(X_n)).
\end{aligned}$$

On the other hand we know by the definition of the map $s_{\mathcal{A}_{\Delta_n}}$, that the elements u_j satisfy the relations

$$\Phi_m(u_0, \ldots, u_m) = \varphi^m(b) = b([\pi^m]_{\phi_1}(X_1), \ldots, [\pi^m]_{\phi_n}(X_n))$$

for every $m \geq 0$. Applying σ we obtain that

$$\begin{aligned}
&\Phi_m(u_0([a_1]_{\phi_1}(X_1), \ldots, [a_n]_{\phi_n}(X_n)), \ldots, u_m([a_1]_{\phi_1}(X_1), \ldots, [a_n]_{\phi_n}(X_n))) \\
&= b([\pi^m]_{\phi_1}([a_1]_{\phi_1}(X_1)), \ldots, [\pi^m]_{\phi_n}([a_n]_{\phi_n}(X_n))) \\
&= b([\pi^m a_1]_{\phi_1}(X_1)), \ldots, [\pi^m a_n]_{\phi_n}(X_n))) \\
&= b([a_1\pi^m]_{\phi_1}(X_1)), \ldots, [a_n\pi^m]_{\phi_n}(X_n))),
\end{aligned}$$

therefore the elements $u_j([a_1]_{\phi_1}(X_1), \dots, [a_n]_{\phi_n}(X_n)) \in \mathcal{A}_{\Delta_n}$ satisfy the required property that determines the elements w_j uniquely. Thus

$$w_m = u_m([a_1]_{\phi_1}(X_1), \dots, [a_n]_{\phi_n}(X_n)) \in \mathcal{A}_{\Delta_n},$$

as desired. $\square$

As for the topology, we now show that if we identify $\mathcal{A}_{\Delta_n}$ with its image in $W(E_{\Delta_n})_L$, then the weak topology of $W(E_{\Delta_n})_L$ induces the weak topology of $\mathcal{A}_{\Delta_n}$ in the sense of Section 2.3. Before we do that, we need one more computational result about the operator φ.

Lemma 3.29. *For $j, k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ we have that*

$$\varphi^k\left((\mathcal{O}_{\Delta_n})_\ell\right) \subseteq (\mathcal{O}_{\Delta_n})_{q^k\ell-(q^k-1)(j-1)} + \pi^j \mathcal{A}_{\Delta_n}.$$

Proof. We will show that

$$\varphi\left((\mathcal{O}_{\Delta_n})_\ell\right) \subseteq (\mathcal{O}_{\Delta_n})_{q\ell-(q-1)(j-1)} + \pi^j \mathcal{A}_{\Delta_n}. \tag{3.22}$$

Let $r_1, \dots, r_n \in \mathbb{Z}$ such that $r_1 + \ldots + r_n = \ell$. If we write $\varphi_i(X_i) = X_i^q + \pi X_i F_i(X_i)$ for some $F_i(X_i) \in \mathcal{O}_L[\![X_i]\!]$, by Lemma 2.6 in the product

$$\varphi(X_1^{r_1} \dots X_n^{r_n}) = \prod_{i \in \Delta_n} \varphi_i(X_i)^{r_i}$$

every factor is either of shape $X_i^q + \pi X_i F_i(X_i)$ or $\displaystyle\sum_{k=0}^{j-1} (-1)^k \frac{\pi^k F_i(X_i)^k}{X_i^{(q-1)k+q}} + \pi^j G_i(X_i)$, depending on the sign of r_i, where $G_i(X_i) \in \mathcal{A}_i$. In the above expansion we either obtain the term $X_1^{qr_1} \dots X_n^{qr_n} \in (\mathcal{O}_{\Delta_n})_{q\ell}$ or a term whose total degree drops by at most $(q-1)$ for every power of π in the chosen factor, therefore (3.22) holds. Iterating (3.22) and using that φ is $\mathcal{O}_L$-linear, we reach the desired conclusion. $\square$

Proposition 3.30. *The map $\mathrm{j}_{\mathcal{A}_{\Delta_n}} : \mathcal{A}_{\Delta_n} \to W(E_{\Delta_n})_L$ is a topological embedding for the weak topologies on both rings.*

Proof. Step 1: Let us show first that the map $\mathrm{j}_{\mathcal{A}_{\Delta_n}}$ is continuous. By Remark 3.26 (iv) it follows that for $\mathbf{x} \in W(E_{\Delta_n})_L$, the sets $\mathbf{x} \boxplus \mathcal{U}_{\ell,m,\mathbf{0}}$ form a fundamental system of open neighbourhoods of $\mathbf{x}$ for $\ell, m \geq 0$. Therefore, it suffices to show that given $\ell, m \geq 0$ we have that

$$\mathrm{j}_{\mathcal{A}_{\Delta_n}}(\mathcal{V}_{\ell+m,m+1}) \subseteq \mathcal{U}_{\ell,m,\mathbf{0}}. \tag{3.23}$$

Let $a \in (\mathcal{O}_{\Delta_n})_{\ell+m}$ and suppose that

$$s_{\mathcal{A}_{\Delta_n}}(a) = \mathbf{b} = (b_0, b_1, \dots) \in W(\mathcal{A}_{\Delta_n})_L.$$

We will show that $b_j \in (\mathcal{O}_{\Delta_n})_\ell + \pi\mathcal{A}_{\Delta_n}$ for all $0 \le j \le m$ by induction on j. For $j = 0$ the claim is clear because $b_0 = a$. Assume that $1 \le j \le m$ and that the claim holds for all $0 \le s \le j-1$. By definition of the map $s_{\mathcal{A}_{\Delta_n}}$, we have that

$$\Phi_j(b_0, \ldots, b_j) = \varphi^j(a)$$

meaning that

$$\pi^j b_j = \varphi^j(a) - \left(b_0^{q^j} + \pi b_1^{q^{j-1}} + \ldots + \pi^{j-1} b_{j-1}^q\right). \tag{3.24}$$

By Lemma 3.29

$$\begin{aligned}\varphi^j(a) &\in (\mathcal{O}_{\Delta_n})_{q^j(\ell+m)-j(q^j-1)} + \pi^{j+1}\mathcal{A}_{\Delta_n} \\ &\subseteq (\mathcal{O}_{\Delta_n})_\ell + \pi^{j+1}\mathcal{A}_{\Delta_n},\end{aligned}$$

where the last inclusion holds because

$$q^j(\ell+m) - j(q^j-1) - \ell = (q^j-1)(\ell+m-j) + m \ge 0,$$

as $j \le m$ and $\ell \ge 0$. By Lemma 1.33 it also follows by our induction hypothesis about b_s for $0 \le s \le j-1$ that

$$\begin{aligned}b_s^{q^{j-s}} &\in (\mathcal{O}_{\Delta_n})_{q^{j-s}\ell} + \pi^{j-s+1}\mathcal{A}_{\Delta_n} \\ &\subseteq (\mathcal{O}_{\Delta_n})_\ell + \pi^{j-s+1}\mathcal{A}_{\Delta_n},\end{aligned}$$

therefore $\pi^s b_s^{q^{j-s}} \in (\mathcal{O}_{\Delta_n})_\ell + \pi^{j+1}\mathcal{A}_{\Delta_n}$. By (3.24) we then know that we can write

$$\pi^j b_j = c + \pi^{j+1} d$$

for some $c \in (\mathcal{O}_{\Delta_n})_\ell$ and $d \in \mathcal{A}_{\Delta_n}$. Therefore $\pi^j(b_j - \pi d) \in (\mathcal{O}_{\Delta_n})_\ell$, hence $b_j - \pi d \in (\mathcal{O}_{\Delta_n})_\ell$ and our induction is complete. It follows that

$$\mathfrak{j}_{\mathcal{A}_{\Delta_n}}\big((\mathcal{O}_{\Delta_n})_{\ell+m}\big) \subseteq \mathcal{U}_{\ell,m,\mathbf{0}}. \tag{3.25}$$

Therefore we have that

$$\begin{aligned}\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(\mathcal{V}_{\ell+m,m+1}) &= \mathfrak{j}_{\mathcal{A}_{\Delta_n}}\big((\mathcal{O}_{\Delta_n})_{\ell+m} + \pi^{m+1}\mathcal{A}_{\Delta_n}\big) \\ &\subseteq \mathfrak{j}_{\mathcal{A}_{\Delta_n}}\big((\mathcal{O}_{\Delta_n})_{\ell+m}\big) \boxplus \pi^{m+1}\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(\mathcal{A}_{\Delta_n}) \\ &\subseteq \mathcal{U}_{\ell,m,\mathbf{0}} \boxplus \pi^{m+1}\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(\mathcal{A}_{\Delta_n}) \\ &\subseteq \mathcal{U}_{\ell,m,\mathbf{0}} \boxplus \pi^{m+1} W(E_{\Delta_n})_L \\ &\subseteq \mathcal{U}_{\ell,m,\mathbf{0}} \boxplus V_{m+1}(E_{\Delta_n})_L \\ &\subseteq \mathcal{U}_{\ell,m,\mathbf{0}},\end{aligned}$$

where the penultimate inclusion holds by Lemma 1.8 (iii) and the last one holds by Lemma 1.7 (i).

Step 2: We now show that $\mathfrak{j}_{\mathcal{A}_{\Delta_n}}$ is open. Since both $(\mathcal{A}_{\Delta_n}, +)$ and $(W(E_{\Delta_n})_L, \boxplus)$ are topological groups it suffices to show that given $\ell \ge 0$, $m \ge 1$ we have that

$$\mathfrak{j}_{\mathcal{A}_{\Delta_n}}^{-1}(\mathcal{U}_{\ell q^{m-1} + m q^{m-1} + (m+1) q^{m-2} + \ldots + (2m-1), 2m-1, \mathbf{0}}) \subseteq \mathcal{V}_{\ell,m}. \tag{3.26}$$

For an element x in the left hand side of (3.26) we have that

$$\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(x) \in \mathcal{U}_{\ell q^{m-1}+mq^{m-1}+(m+1)q^{m-2}+\ldots+(2m-1),2m-1,\mathbf{0}}.$$

Because

$$\Phi_0 \circ \mathfrak{j}_{\mathcal{A}_{\Delta_n}}(x) = x \bmod \pi\mathcal{A}_{\Delta_n}$$

we have that

$$x = y_1 + \pi x_1$$

for some $y_1 \in (\mathcal{O}_{\Delta_n})_{\ell q^{m-1}+mq^{m-1}+(m+1)q^{m-2}+\ldots+(2m-1)}$ and $x_1 \in \mathcal{A}_{\Delta_n}$. We have that

$$\pi\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(x_1) = \mathfrak{j}_{\mathcal{A}_{\Delta_n}}(x) \boxminus \mathfrak{j}_{\mathcal{A}_{\Delta_n}}(y_1).$$

We know that

$$\begin{aligned} \mathfrak{j}_{\mathcal{A}_{\Delta_n}}(x) &\in \mathcal{U}_{\ell q^{m-1}+mq^{m-1}+(m+1)q^{m-2}+\ldots+(2m-1),2m-1,\mathbf{0}} \\ &\subseteq \mathcal{U}_{\ell q^{m-1}+mq^{m-1}+(m+1)q^{m-2}+\ldots+(2m-2)q,2m-1,\mathbf{0}} \end{aligned}$$

by our assumption about x, while

$$\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(y_1) \in \mathcal{U}_{\ell q^{m-1}+mq^{m-1}+(m+1)q^{m-2}+\ldots+(2m-2)q,2m-1,\mathbf{0}}$$

by (3.25) since $y_1 \in (\mathcal{O}_{\Delta_n})_{\ell q^{m-1}+mq^{m-1}+(m+1)q^{m-2}+\ldots+(2m-1)}$. The set

$$\mathcal{U}_{\ell q^{m-1}+mq^{m-1}+(m+1)q^{m-2}+\ldots+(2m-2)q,2m-1,\mathbf{0}}$$

is closed under addition and additive inverses by (3.20) and (3.21), therefore

$$\pi\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(x_1) \in \mathcal{U}_{\ell q^{m-1}+mq^{m-1}+(m+1)q^{m-2}+\ldots+(2m-2)q,2m-1,\mathbf{0}}.$$

By Lemma 1.8 (ii), it follows that

$$\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(x_1) \in \mathcal{U}_{\ell q^{m-2}+mq^{m-2}+(m+1)q^{m-3}+\ldots+(2m-2),2m-2,\mathbf{0}}.$$

Continuing in the same manner, we will obtain elements $x_1, \ldots, x_m, y_1, \ldots, y_m$ such that

$$x_{j-1} = y_j + \pi x_j,$$

$$y_j \in (\mathcal{O}_{\Delta_n})_{\ell q^{m-j}+mq^{m-j}+(m+1)q^{m-j-1}+\ldots+(2m-j)} \subseteq (\mathcal{O}_{\Delta_n})_\ell$$

and $x_j \in \mathcal{A}_{\Delta_n}$ for $1 \le j \le m$, where $x_0 = x$. Then

$$\begin{aligned} x &= y_1 + \pi x_1 \\ &= y_1 + \pi(y_2 + \pi x_2) \\ &= \ldots \\ &= y_1 + \pi y_2 \ldots + \pi^{m-1} y_m + \pi^m x_m \in (\mathcal{O}_{\Delta_n})_\ell + \pi^m \mathcal{A}_{\Delta_n} = \mathcal{V}_{\ell,m}, \end{aligned}$$

as desired. □

3.5 The ring $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ and its embedding into $W(\mathbb{C}^\flat_{p,\Delta_n})_L$

Let $\mathcal{B}_L$ denote the field of fractions of $\mathcal{A}_L$. We also let $\mathcal{B}_L^{\mathrm{ur}}$ denote the maximal unramified extension of $\mathcal{B}_L$ with ring of integers $\mathcal{A}_L^{\mathrm{ur}}$. Since E_L is the residue field of $\mathcal{A}_L$, by Proposition 1.2.6 of [Sch17] we know that $\mathcal{A}_L^{\mathrm{ur}}$ is a DVR with uniformizer π and residue field E_L^{sep}, and that we have a natural isomorphism of Galois groups

$$\mathrm{Gal}(\mathcal{B}_L^{\mathrm{ur}}/\mathcal{B}_L) \simeq \mathrm{Gal}(E_L^{\mathrm{sep}}/E_L). \tag{3.27}$$

To define the functors between our categories we introduce a multivariable analog of $\mathcal{A}_L^{\mathrm{ur}}$. For $i \in \Delta_n$, we let $\mathcal{B}_i$ be the fraction field of $\mathrm{Frac}(\mathcal{A}_i)$, while by $\mathcal{B}_i^{\mathrm{ur}}$ we denote the maximal unramified extension of $\mathcal{B}_i$ and by $\mathcal{A}_i^{\mathrm{ur}}$ we denote the ring of integers of $\mathcal{B}_i^{\mathrm{ur}}$. We define

$$\mathcal{A}_{\Delta_n,\circ} \coloneqq \mathcal{A}_1 \otimes_{\mathcal{O}_L} \ldots \otimes_{\mathcal{O}_L} \mathcal{A}_n,$$
$$\mathcal{A}^{\mathrm{ur}}_{\Delta_n,\circ} \coloneqq \mathcal{A}_1^{\mathrm{ur}} \otimes_{\mathcal{O}_L} \ldots \otimes_{\mathcal{O}_L} \mathcal{A}_n^{\mathrm{ur}}$$

and

$$\mathcal{A}^{\mathrm{ur}}_{\Delta_n} \coloneqq \mathcal{A}^{\mathrm{ur}}_{\Delta_n,\circ} \otimes_{\mathcal{A}_{\Delta_n,\circ}} \mathcal{A}_{\Delta_n}.$$

For $i \in \Delta_n$, the ring $\mathcal{A}_i^{\mathrm{ur}}$ is a DVR with uniformizer π and residue field $\mathcal{A}_i^{\mathrm{ur}}/\pi\mathcal{A}_i^{\mathrm{ur}} \simeq E_i^{\mathrm{sep}}$, therefore

$$\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi\mathcal{A}^{\mathrm{ur}}_{\Delta_n} \simeq E^{\mathrm{sep}}_{\Delta_n,\circ} \otimes_{E_{\Delta_n,\circ}} E_{\Delta_n} = E^{\mathrm{sep}}_{\Delta_n}.$$

We also let

$$\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \coloneqq \varprojlim_m \mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$$

denote the π-adic completion of $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$.

Note that

$$\mathcal{A}^{\mathrm{ur}}_{\Delta_n} = \varinjlim\, (\mathcal{A}_1' \otimes_{\mathcal{O}_L} \ldots \otimes_{\mathcal{O}_L} \mathcal{A}_n') \otimes_{\mathcal{A}_{\Delta_n,\circ}} \mathcal{A}_{\Delta_n}$$

where the colimit is over all collections of rings of integers $\mathcal{A}_i'$ of $\mathcal{B}_i'$, where $\mathcal{B}_i'$ is a finite unramified extension of $\mathcal{B}_i$, for $i \in \Delta_n$. We denote

$$\mathcal{A}'_{\Delta_n,\circ} \coloneqq \mathcal{A}_1' \otimes_{\mathcal{O}_L} \ldots \otimes_{\mathcal{O}_L} \mathcal{A}_n'$$

and

$$\mathcal{A}'_{\Delta_n} \coloneqq \mathcal{A}'_{\Delta_n,\circ} \otimes_{\mathcal{A}_{\Delta_n,\circ}} \mathcal{A}_{\Delta_n}.$$

Lemma 3.31. *(i) $\mathcal{A}'_{\Delta_n,\circ}$ is a finite free $\mathcal{A}_{\Delta_n,\circ}$-module.*

(ii) $\mathcal{A}'_{\Delta_n}$ is a finite free $\mathcal{A}_{\Delta_n}$-module.

(iii) The natural map

$$\mathcal{A}_{\Delta_n} \hookrightarrow \mathcal{A}^{\mathrm{ur}}_{\Delta_n}$$
$$a \mapsto (1 \otimes \ldots \otimes 1) \otimes a$$

is an embedding.

(iv) The natural map

$$\begin{aligned} \mathcal{A}'_{\Delta_n} &\hookrightarrow \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \\ (a_1 \otimes \ldots \otimes a_n) \otimes a &\mapsto (a_1 \otimes \ldots \otimes a_n) \otimes a \end{aligned}$$

is an embedding, where $a_i \in \mathcal{A}'_i$ for $i \in \Delta_n$ and $a \in \mathcal{A}_{\Delta_n}$.

Proof. Each $\mathcal{A}'_i$ is a finite free $\mathcal{A}_i$-module by Lemma 1.2.4 (ii) of [Sch17] and admits a basis over $\mathcal{A}_i$ containing the element 1, therefore $\mathcal{A}'_{\Delta_n,\circ}$ is a finite free $\mathcal{A}_{\Delta_n,\circ}$-module admitting a basis containing the element $1 \otimes \ldots \otimes 1$, which in particular proves the claim of (i). Hence $\mathcal{A}'_{\Delta_n}$ is also a finite free $\mathcal{A}_{\Delta_n}$-module admitting a basis containing the element $(1 \otimes \ldots \otimes 1) \otimes 1$. This implies that the natural map

$$\begin{aligned} \mathcal{A}_{\Delta_n} &\hookrightarrow \mathcal{A}'_{\Delta_n} \\ a &\mapsto (1 \otimes \ldots \otimes 1) \otimes a \end{aligned}$$

is an embedding. Therefore the result of (iii) follows using a colimit argument. Considering another collection of finite unramified extensions $(\mathcal{B}''_i/\mathcal{B}'_i)_{i\in\Delta_n}$ with rings of integers $\mathcal{A}''_i$, a similar argument shows that $\mathcal{A}''_{\Delta_n}$ is a finite free $\mathcal{A}'_{\Delta_n}$-module admitting a basis containing the element $(1 \otimes \ldots \otimes 1) \otimes 1$. Therefore the map

$$\begin{aligned} \mathcal{A}'_{\Delta_n} &\hookrightarrow \mathcal{A}''_{\Delta_n} \\ (a_1 \otimes \ldots \otimes a_n) \otimes a &\mapsto (a_1 \otimes \ldots \otimes a_n) \otimes a \end{aligned}$$

is an embedding and a colimit argument proves (iv). $\square$

An immediate corollary to Lemma 3.31 shows that the rings $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$ and $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ have some useful flatness properties.

Corollary 3.32. *(i) $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$ is a flat $\mathcal{O}_L$-algebra.*

(ii) $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ is a flat $\mathcal{O}_L$-algebra.

(iii) $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ is a flat $\mathcal{A}_{\Delta_n}$-algebra.

Proof. (i) It suffices to show that $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$ is torsion-free over $\mathcal{O}_L$, for which it is enough to prove that $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}[\pi] = 0$, in other words, that the sequence

$$0 \longrightarrow \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \xrightarrow{\ \pi\ } \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \tag{3.28}$$

is exact. The sequence

$$0 \longrightarrow \mathcal{A}_{\Delta_n} \xrightarrow{\ \pi\ } \mathcal{A}_{\Delta_n} \tag{3.29}$$

is clearly exact. By Lemma 3.31 (i) $\mathcal{A}^{\mathrm{ur}}_{\Delta_n,\circ}$ is a flat $\mathcal{A}_{\Delta_n,\circ}$-module, being a colimit of flat modules. Applying $\mathcal{A}^{\mathrm{ur}}_{\Delta_n,\circ} \otimes_{\mathcal{A}_{\Delta_n,\circ}} -$ to (3.29), we get that (3.28) is exact.

(ii) It suffices to show that $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}[\pi]=0$. Let

$$a=(a_m \bmod \pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n})_{m\geq 1}\in\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$$

for some $a_m\in\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$, be such that $\pi a=0$. Then for every $m\geq 1$ we have that $\pi a_{m+1}\in\pi^{m+1}\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$. By part (i), it follows that $a_{m+1}\in\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$, therefore $a_m\in\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$. This means that $a=0$, as desired.

(iii) $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$ is a flat $\mathcal{A}_{\Delta_n}$-module, being the colimit of the free $\mathcal{A}_{\Delta_n}$-modules $\mathcal{A}'_{\Delta_n}$. Therefore $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ is flat over $\mathcal{A}_{\Delta_n}$ by Corollary 1.50 due to the Noetherianity of $\mathcal{A}_{\Delta_n}$. □

We will extend the embedding of $\mathcal{A}_{\Delta_n}$ into $W(\mathbb{C}^\flat_{p,\Delta_n})_L$ to an embedding of $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ into $W(\mathbb{C}^\flat_{p,\Delta_n})_L$. The following lemma shows that the Frobenius map φ_i on $\mathcal{A}_i$ can be extended to rings of integers of unramified extensions of $\mathcal{B}_i$.

Lemma 3.33. *Let $i\in\Delta_n$ and $\mathcal{B}'_i$ be a finite unramified extension of $\mathcal{B}_i$ with ring of integers $\mathcal{A}'_i$. The Frobenius map φ_i on $\mathcal{A}_i$ has a unique extension to an endomorphism σ_i of $\mathcal{A}'_i$ such that $\sigma_i(a)\equiv a^q \bmod \pi\mathcal{A}'_i$ for any $a\in\mathcal{A}'_i$.*

Proof. See Lemma 3.1.2 in [Sch17]. □

Taking colimits, Lemma 3.33 shows that φ_i extends uniquely to an endomorphism of $\mathcal{A}^{\mathrm{ur}}_i$, which we simply denote by Fr_i, for which

$$\mathrm{Fr}_i(a)\equiv a^q \bmod \pi\mathcal{A}^{\mathrm{ur}}_i$$

for every $a\in\mathcal{A}^{\mathrm{ur}}_i$. On $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$, we obtain the maps

$$\begin{aligned}\varphi_i:\mathcal{A}^{\mathrm{ur}}_{\Delta_n}&\to\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\\ (a_1\otimes\ldots\otimes a_n)\otimes a&\mapsto(a_1\otimes\ldots\otimes\mathrm{Fr}_i(a_i)\otimes\ldots\otimes a_n)\otimes\varphi_i(a)\end{aligned}$$

for $i\in\Delta_n$ and the map

$$\begin{aligned}\varphi_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}:\mathcal{A}^{\mathrm{ur}}_{\Delta_n}&\to\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\\ (a_1\otimes\ldots\otimes a_n)\otimes a&\mapsto(\mathrm{Fr}_1(a_1)\otimes\ldots\otimes\mathrm{Fr}_n(a_n))\otimes\varphi(a)\end{aligned}$$

where $a_i\in\mathcal{A}^{\mathrm{ur}}_i$ and $a\in\mathcal{A}_{\Delta_n}$. Because φ_i is $\mathcal{O}_L$-linear on $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$, it extends to a map on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$, which we also call φ_i. The element π is not a zero divisor in $\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$ and we also have that

$$\varphi_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}(a)\equiv a^q \bmod \pi\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$$

for all $a\in\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$. Therefore the conditions of Proposition 1.10 are satisfied and we have a map

$$s_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}:\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\to W(\mathcal{A}^{\mathrm{ur}}_{\Delta_n})_L.$$

We define $\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} : \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \to W(\mathbb{C}_{p,\Delta_n})_L$ to be the composition

$$\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} : \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \xrightarrow{s_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}} W(\mathcal{A}^{\mathrm{ur}}_{\Delta_n})_L \xrightarrow{W(\mathrm{pr})_L} W(E^{\mathrm{sep}}_{\Delta_n})_L \subseteq W(\mathbb{C}^{\flat}_{p,\Delta_n})_L,$$

where

$$\mathrm{pr} : \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \to \mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi\mathcal{A}^{\mathrm{ur}}_{\Delta_n} \simeq E^{\mathrm{sep}}_{\Delta_n}$$

is the natural projection map. The map $\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ clearly extends the map $\mathfrak{j}_{\mathcal{A}_{\Delta_n}}$ from the previous section. We now show that $\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ is an embedding as well.

Lemma 3.34. *The map $\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} : \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \to W(E^{\mathrm{sep}}_{\Delta_n})_L$ is an embedding.*

Proof. Fix a collection $(\mathcal{B}'_i/\mathcal{B}_i)_{i\in\Delta_n}$ of finite unramified extensions and let $\mathcal{A}'_i$ denote the ring of integers of $\mathcal{B}'_i$ for $i \in \Delta_n$. The map $\varphi_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ restricts to a map on $\mathcal{A}'_{\Delta_n}$, which we denote by $\varphi_{\mathcal{A}'_{\Delta_n}}$. Therefore $\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ restricts to a map

$$\mathfrak{j}_{\mathcal{A}'_{\Delta_n}} : \mathcal{A}'_{\Delta_n} \to W(E'_{\Delta_n})_L$$

where $E'_i \simeq \mathcal{A}'_i/\pi\mathcal{A}'_i$ is a finite separable extension of E_i for $i \in \Delta_n$. Let $b \in \ker \mathfrak{j}_{\mathcal{A}'_{\Delta_n}}$. By the definition of the map $s_{\mathcal{A}'_{\Delta_n}}$ we have

$$\Phi_0 \circ \mathfrak{j}_{\mathcal{A}'_{\Delta_n}}(b) = b \bmod \pi\mathcal{A}'_{\Delta_n},$$

therefore $b \in \pi\mathcal{A}'_{\Delta_n}$. Write $b = \pi b_1$ for some $b_1 \in \mathcal{A}'_{\Delta_n}$. By Proposition 3.6, $\mathbb{C}^{\flat}_{p,\Delta_n}$ is reduced, therefore $\pi \cdot \mathbf{1}_{W(\mathbb{C}^{\flat}_{p,\Delta_n})_L}$ is not a zero divisor in $W(\mathbb{C}^{\flat}_{p,\Delta_n})_L$. By our assumption about b we know that $\pi\mathfrak{j}_{\mathcal{A}'_{\Delta_n}}(b_1) = \mathfrak{j}_{\mathcal{A}'_{\Delta_n}}(b) = 0$, therefore

$$\mathfrak{j}_{\mathcal{A}'_{\Delta_n}}(b_1) = 0$$

and by the reasoning above $b_1 \in \pi\mathcal{A}'_{\Delta_n}$. Continuing in the same manner, we obtain that

$$b \in \bigcap_{m\geq 1} \pi^m\mathcal{A}'_{\Delta_n} = 0.$$

The last equality holds because $\mathcal{A}'_{\Delta_n}$ is π-adically complete being a finitely generated module over the π-adically complete Noetherian ring $\mathcal{A}_{\Delta_n}$. From a colimit argument we can conclude that $\ker \mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} = 0$, as desired. $\square$

The map $\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ is an $\mathcal{O}_L$-algebra endomorphism, therefore it induces a map

$$\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}} : \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \to \varprojlim_{m\geq 1} W(E^{\mathrm{sep}}_{\Delta_n})_L/\pi^m W(E^{\mathrm{sep}}_{\Delta_n})_L \simeq W(E^{\mathrm{sep}}_{\Delta_n})_L$$

where the last isomorphism follows from Lemma 1.8 (iv).

Lemma 3.35. *The map $\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}} : \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \to W(E^{\mathrm{sep}}_{\Delta_n})_L$ is an embedding.*

Proof. Let $b = (b_m)_{m\geq 1} \in \varprojlim_{m\geq 1} \mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$ be such that

$$\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}(b) = 0$$

where $b_m \in \mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}$ for $m \geq 1$. By our definition of $\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}$, we know that

$$\Phi_0 \circ \mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}(b) = b_1.$$

Therefore $b_1 = 0$ and thus $b \in \pi\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$, allowing us to write $b = \pi\widetilde{b}$ for some $\widetilde{b} \in \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$. By Proposition 3.6, $\mathbb{C}^{\flat}_{p,\Delta_n}$ is reduced, therefore $\pi \cdot \mathbf{1}_{W(\mathbb{C}^{\flat}_{p,\Delta_n})_L}$ is not a zero divisor in $W(\mathbb{C}^{\flat}_{p,\Delta_n})_L$. Since $\pi\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}(\widetilde{b}) = \mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}(b) = 0$, we have that

$$\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}(\widetilde{b}) = 0$$

and by the above argument $\widetilde{b} \in \pi\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$. Continuing in the same manner, we obtain that

$$b \in \bigcap_{m\geq 1} \pi^m\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} = 0.$$

The last equality holds because $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ is π-adically complete by [Sta18, Tag 05GG]. □

Lemma 3.36. *The group $G_{\Delta_n,L}$ preserves the subset $\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}) \subseteq W(\mathbb{C}^{\flat}_{p,\Delta_n})_L$, in other words*

$$\sigma\left(\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}})\right) \subseteq \mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}})$$

for every $\sigma \in G_{\Delta_n,L}$.

Proof. Since every $\sigma \in G_{\Delta_n,L}$ acts $\mathcal{O}_L$-linearly and $\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}$ is an $\mathcal{O}_L$-linear map, it suffices to show that $G_{\Delta_n,L}$ preserves $\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}(\mathcal{A}^{\mathrm{ur}}_{\Delta_n})$. Let $x \in \mathcal{A}^{\mathrm{ur}}_{\Delta_n}$ and $\sigma \in G_{\Delta_n,L}$. To show that $\sigma(\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}(x)) \in \mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}(\mathcal{A}^{\mathrm{ur}}_{\Delta_n})$, by linearity it suffices to consider elements of the form

$$x = (a_1 \otimes \ldots \otimes a_n) \otimes a \in \mathcal{A}^{\mathrm{ur}}_{\Delta_n}$$

where $a_i \in \mathcal{A}^{\mathrm{ur}}_i$ for $i \in \Delta_n$ and $a \in \mathcal{A}_{\Delta_n}$. Writing

$$\sigma(\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}(x)) = \sigma\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}((1 \otimes \ldots \otimes 1) \otimes a) \prod_{i\in\Delta_n} \sigma\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}((1 \otimes \ldots \otimes a_i \otimes \ldots 1) \otimes 1)$$

it suffices to show that each of the above factors is in $\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}(\mathcal{A}^{\mathrm{ur}}_{\Delta_n})$. By Lemma 3.31 (ii), we can regard $\sigma\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}((1 \otimes \ldots \otimes 1) \otimes a)$ simply as $\sigma\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(a)$ which by Lemma 3.28 (ii) equals $\mathfrak{j}_{\mathcal{A}_{\Delta_n}}(\sigma(a))$. In other words

$$\sigma\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}((1 \otimes \ldots \otimes 1) \otimes a) = \mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}((1 \otimes \ldots \otimes 1) \otimes \sigma(a)).$$

We now proceed to the other factors. To simplify notation, for $y \in \mathcal{A}^{\mathrm{ur}}_i$ we write

$$\iota_i(y) \coloneqq (1 \otimes \ldots \otimes y \otimes \ldots \otimes 1) \otimes 1 \in \mathcal{A}^{\mathrm{ur}}_{\Delta_n}$$

where in $(1\otimes\ldots\otimes y\otimes\ldots\otimes 1)$ the i-th component is y and the other components are 1. For $i\in\Delta_n$, we also let

$$\begin{aligned}\bar{\iota}_i : E_i^{\mathrm{sep}} &\to E_{\Delta_n,\circ}^{\mathrm{sep}}\otimes_{E_{\Delta_n,\circ}} E_{\Delta_n}\\ e &\mapsto (1\otimes\ldots\otimes e\otimes\ldots\otimes 1)\otimes 1\end{aligned}$$

for $e\in E_i^{\mathrm{sep}}$ and $\mathrm{j}_{\mathcal{A}_i^{\mathrm{ur}}}$ be the composition of the maps

$$\mathcal{A}_i^{\mathrm{ur}} \xrightarrow{s_{\mathcal{A}_i^{\mathrm{ur}}}} W(\mathcal{A}_i^{\mathrm{ur}})_L \xrightarrow{W(\mathrm{pr})_L} W(E_i^{\mathrm{sep}})_L,$$

where $s_{\mathcal{A}_i^{\mathrm{ur}}}$ is the map from Proposition 1.10 applied for $B=\mathcal{A}_i^{\mathrm{ur}}$ and $\psi=\mathrm{Fr}_i$. We also write $\sigma=\prod_{i\in\Delta_n}\sigma_i$ for $\sigma_i\in G_{i,L}$. We claim that the diagram

$$\begin{tikzcd}
\mathcal{A}_i^{\mathrm{ur}} \arrow[r, "s_{\mathcal{A}_i^{\mathrm{ur}}}"] \arrow[d, "\iota_i"] & W(\mathcal{A}_i^{\mathrm{ur}})_L \arrow[d, "W(\iota_i)_L"] \\
\mathcal{A}_{\Delta_n}^{\mathrm{ur}} \arrow[r, "s_{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}"] & W(\mathcal{A}_{\Delta_n}^{\mathrm{ur}})_L
\end{tikzcd} \tag{3.30}$$

commutes. Indeed, let $y\in\mathcal{A}_i^{\mathrm{ur}}$ and suppose that

$$s_{\mathcal{A}_i^{\mathrm{ur}}}(y)=(z_0,z_1,\ldots)\in W(\mathcal{A}_i^{\mathrm{ur}})_L.$$

By the definition of $s_{\mathcal{A}_i^{\mathrm{ur}}}$ and the fact that π is not a zero divisor in $\mathcal{A}_i^{\mathrm{ur}}$, we know that the elements $z_0,z_1,\ldots$ are determined uniquely by their requirement to satisfy

$$\Phi_m(z_0,\ldots,z_m)=\mathrm{Fr}_i^m(y)$$

for all $m\geq 0$. Therefore

$$\Phi_m(\iota_i(z_0),\ldots,\iota_i(z_m))=\iota_i(\Phi_m(z_0,\ldots,z_m))=\iota_i(\mathrm{Fr}_i^m(y))=\varphi^m_{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}(\iota_i(y)),$$

thus by the definition of $s_{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$, we have that

$$s_{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}(\iota_i(y))=W(\iota_i)_L\circ s_{\mathcal{A}_i^{\mathrm{ur}}}(y),$$

meaning that (3.30) commutes. We also have that

$$\begin{tikzcd}
W(\mathcal{A}_i^{\mathrm{ur}})_L \arrow[r, "W(\mathrm{pr})_L"] \arrow[d, "W(\iota_i)_L"] & W(E_i^{\mathrm{sep}})_L \arrow[d, "W(\bar{\iota}_i)_L"] \\
W(\mathcal{A}_{\Delta_n}^{\mathrm{ur}})_L \arrow[r, "W(\mathrm{pr})_L"] & W(E_{\Delta_n}^{\mathrm{sep}})_L
\end{tikzcd} \tag{3.31}$$

commutes, where pr is the projection modulo π. Finally, we also have that

$$\begin{tikzcd}
W(E_i^{\mathrm{sep}})_L \arrow[r, "W(\sigma_i)_L"] \arrow[d, "W(\bar{\iota}_i)_L"] & W(E_i^{\mathrm{sep}})_L \arrow[d, "W(\bar{\iota}_i)_L"] \\
W(E_{\Delta_n}^{\mathrm{sep}})_L \arrow[r, "W(\sigma)_L"] & W(E_{\Delta_n}^{\mathrm{sep}})_L
\end{tikzcd} \tag{3.32}$$

commutes as well, because $\bar{\iota}_i(\sigma_i(e)) = \sigma(\bar{\iota}_i(e))$ for $e \in E_i^{\mathrm{sep}}$. Therefore

$$\begin{aligned}\sigma \mathfrak{j}_{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}(\iota_i(y)) &= \sigma \circ W(\mathrm{pr})_L \circ s_{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}(\iota_i(y)) \\ &= \sigma \circ W(\mathrm{pr})_L \circ W(\iota_i)_L \circ s_{\mathcal{A}_i^{\mathrm{ur}}}(y) \\ &= \sigma \circ W(\bar{\iota}_i)_L \circ W(\mathrm{pr})_L \circ s_{\mathcal{A}_i^{\mathrm{ur}}}(y) \\ &= W(\bar{\iota}_i)_L \circ \sigma_i \circ W(\mathrm{pr})_L \circ s_{\mathcal{A}_i^{\mathrm{ur}}}(y) \\ &= W(\bar{\iota}_i)_L \circ \sigma_i \circ \mathfrak{j}_{\mathcal{A}_i^{\mathrm{ur}}}(y)\end{aligned}$$

where in the second line we used the commutativity in (3.30), in the third line the commutativity of (3.31) and in the fourth line the commutativity of (3.32). By the proof of Lemma 3.1.3 in [Sch17] we know that

$$\sigma_i \circ \mathfrak{j}_{\mathcal{A}_i^{\mathrm{ur}}}(y) = \mathfrak{j}_{\mathcal{A}_i^{\mathrm{ur}}}(w)$$

for a unique $w \in \mathcal{A}_i^{\mathrm{ur}}$. Therefore

$$\begin{aligned}\sigma \mathfrak{j}_{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}(\iota_i(y)) &= W(\bar{\iota}_i)_L \circ \mathfrak{j}_{\mathcal{A}_i^{\mathrm{ur}}}(w) \\ &= W(\bar{\iota}_i)_L \circ W(\mathrm{pr})_L \circ s_{\mathcal{A}_i^{\mathrm{ur}}}(w) \\ &= W(\mathrm{pr})_L \circ W(\iota_i)_L \circ s_{\mathcal{A}_i^{\mathrm{ur}}}(w) \\ &= W(\mathrm{pr})_L \circ s_{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}(\iota_i(w)) \\ &= \mathfrak{j}_{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}(\iota_i(w))\end{aligned}$$

where the equality in the third line holds by the commutativity in (3.31) and the equality in the fourth line holds by the commutativity of (3.30). □

Remark 3.37. (i) Using Lemma 3.35, we can identify $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ with a subspace of $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$. We equip $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ with the subspace topology of the weak topology of $W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ and call this the weak topology on $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$. By Lemma 3.25 it follows that $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ is a topological ring for the weak topology. Therefore, the theory of Section 1.4 applies for $R = \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ and we call the linear topology on a finitely generated $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$-module the weak topology as well.

(ii) From Remark 3.26 (iii) we conclude that

$$\left(\mathfrak{U}_{\varepsilon,m,\mathbf{0}} \cap \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}\right)_{0\leq\varepsilon\leq 1, m\in\mathbb{N}_{\geq 0}}$$

is a system of open neighbourhoods closed under addition.

(iii) Lemma 3.36 shows that $G_{\Delta_n,L}$ acts on $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ and the action is continuous by Lemma 3.24. In the one variable case we recover the fact that $G_{i,L}$ acts on $\mathcal{A}_i^{\mathrm{ur}}$ and the proof of Lemma 3.36 shows that the relation between this action and the one in the multivariable case is given by the formula

$$\sigma((a_1 \otimes \ldots \otimes a_n) \otimes a) = (\sigma_1(a_1) \otimes \ldots \otimes \sigma_n(a_n)) \otimes \sigma(a), \tag{3.33}$$

where $a_i \in \mathcal{A}_i^{\mathrm{ur}}$ for $i \in \Delta_n, a \in \mathcal{A}_{\Delta_n}$ and $\sigma = \prod_{i\in\Delta_n} \sigma_i$ for $\sigma_i \in G_{i,L}$.

3.5.1 The embedding modulo π^m

So far we managed to establish that the map $\mathfrak{j}_{\mathcal{A}_{\Delta_n}}$ is a topological embedding by Lemma 3.30, while the map $\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}$ is a topological embedding by definition. In other words, the inclusions induced by $\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}$

$$\mathcal{A}_{\Delta_n} \hookrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \hookrightarrow W(\mathbb{C}^\flat_{p,\Delta_n})_L$$

are topological embeddings for the weak topology on each of the rings. Note that restricting $\mathfrak{j}_{\mathcal{A}_{\Delta_n}}$ to $\mathcal{O}_L$ gives us the map $\mathcal{O}_L \to W(\kappa_L)_L$ from Lemma 1.11, which is an isomorphism. Therefore $\mathfrak{j}_{\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}}$ induces the inclusions

$$\mathcal{O}_L \hookrightarrow \mathcal{A}_{\Delta_n} \hookrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \hookrightarrow W(\mathbb{C}^\flat_{p,\Delta_n})_L. \tag{3.34}$$

We will now show that the composition of the first two arrows in (3.34) is a topological embedding for the π-adic topology on $\mathcal{O}_L$, together with a similar statement when we take quotients by a power of π.

Lemma 3.38. *(i) The weak topology on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ induces the π-adic topology on $\mathcal{O}_L$.*

(ii) The natural map

$$\mathcal{O}_L/\pi^m\mathcal{O}_L \to \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}/\pi^m\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$$

is an embedding and the weak topology on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}/\pi^m\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ induces the π-adic topology on $\mathcal{O}_L/\pi^m\mathcal{O}_L$ for $m \geq 1$.

Proof. (i) By what we explained above it suffices to show that the weak topology on $\mathcal{A}_{\Delta_n}$ induces the π-adic topology on $\mathcal{O}_L$. This is easily observed from the equality

$$\begin{aligned}\mathcal{V}_{\ell,m} \cap \mathcal{O}_L &= ((\mathcal{O}_{\Delta_n})_\ell + \pi^m\mathcal{A}_{\Delta_n}) \cap \mathcal{O}_L \\ &= \pi^m\mathcal{O}_L\end{aligned}$$

for $\ell, m \geq 1$.

(ii) We will first show that the natural maps

$$\mathcal{O}_L/\pi^m\mathcal{O}_L \to W(\mathbb{C}^\flat_{p,\Delta_n})_L/\pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L$$

and

$$\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}/\pi^m\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \to W(\mathbb{C}^\flat_{p,\Delta_n})_L/\pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L$$

are injective. For the first one, note that since $\mathcal{O}_L$ is a DVR with uniformizer π, the kernel of the natural map

$$\mathcal{O}_L \to W(\mathbb{C}^\flat_{p,\Delta_n})_L/\pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L$$

is an ideal of $\mathcal{O}_L$ generated by a power of π containing $\pi^m\mathcal{O}_L$. By Lemma 1.9 (ii) and Lemma 1.7 (i)

$$\pi^{m-1} \boxplus \pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L = (0, \ldots, 0, 1, 0, \ldots) \boxplus V_m(\mathbb{C}^\flat_{p,\Delta_n})_L \neq \mathbf{0} \boxplus V_m(\mathbb{C}^\flat_{p,\Delta_n})_L,$$

implying that the kernel equals $\pi^m \mathcal{O}_L$. For the other map, note first that we have an isomorphism

$$\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}/\pi^m \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \simeq \mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m \mathcal{A}^{\mathrm{ur}}_{\Delta_n},$$

therefore it suffices to show that $\mathfrak{j}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ induces an inclusion

$$\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \hookrightarrow W(\mathbb{C}^\flat_{p,\Delta_n})_L/\pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L = W(\mathbb{C}^\flat_{p,\Delta_n})_L/V_m(\mathbb{C}^\flat_{p,\Delta_n})_L \simeq W_m(\mathbb{C}^\flat_{p,\Delta_n})_L.$$

We do so by induction on m. For $m = 1$, using that $\Phi_0 \circ s_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} = \mathrm{id}_{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$, our map is

$$\begin{aligned}\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi \mathcal{A}^{\mathrm{ur}}_{\Delta_n} &\to W_1(\mathbb{C}^\flat_{p,\Delta_n})_L \\ a &\mapsto (a, 0, \dots) \boxplus V_1(\mathbb{C}^\flat_{p,\Delta_n})_L\end{aligned}$$

which is clearly injective. Let $m > 1$ and assume that we showed that the injectivity holds for all $1 \leq j \leq m-1$. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \pi^{m-1}\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \arrow[r] \arrow[d] & \mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \arrow[r] \arrow[d] & \mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^{m-1} \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^{m-1}W(\mathbb{C}^\flat_{p,\Delta_n})_L/\pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L \arrow[r] & W_m(\mathbb{C}^\flat_{p,\Delta_n})_L \arrow[r] & W_{m-1}(\mathbb{C}^\flat_{p,\Delta_n})_L \arrow[r] & 0.
\end{tikzcd}$$

As π is not a zero divisor in $W(\mathbb{C}^\flat_{p,\Delta_n})_L$ by Proposition 3.6, it follows that multiplication by π^{m-1} induces the isomorphisms

$$\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi \mathcal{A}^{\mathrm{ur}}_{\Delta_n} \simeq \pi^{m-1}\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m \mathcal{A}^{\mathrm{ur}}_{\Delta_n}$$

and

$$W(\mathbb{C}^\flat_{p,\Delta_n})_L/\pi W(\mathbb{C}^\flat_{p,\Delta_n})_L \simeq \pi^{m-1}W(\mathbb{C}^\flat_{p,\Delta_n})_L/\pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L,$$

therefore the left vertical arrow of the above diagram is an injection by what we proved for $m = 1$. By our induction hypothesis, the right vertical arrow of the above diagram is an injection as well, therefore the middle vertical arrow is an injection, as desired.

Coming back to our goal, we observe that the π-adic topology on $\mathcal{O}_L/\pi^m\mathcal{O}_L$ is the discrete topology. Therefore, it suffices to show that there exists an open neighbourhood of zero $U \subseteq \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}/\pi^m \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ such that

$$U \cap (\mathcal{O}_L/\pi^m\mathcal{O}_L) = 0.$$

The topologies of $\mathcal{O}_L/\pi^m\mathcal{O}_L$, $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}/\pi^m \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ and $W(\mathbb{C}^\flat_{p,\Delta_n})_L/\pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L$ are defined using the quotient topology, therefore we know by part (i) that the maps

$$\mathcal{O}_L/\pi^m\mathcal{O}_L \hookrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}/\pi^m \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \hookrightarrow W(\mathbb{C}^\flat_{p,\Delta_n})_L/\pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L$$

are continuous. Consider the open neighbourhood $\mathfrak{U}_{\delta,m-1,\mathbf{0}} \subseteq W(\mathbb{C}^\flat_{p,\Delta_n})_L$ where $0 < \delta < 1$ is some fixed constant. Then

$$\pi^m W(\mathbb{C}^\flat_{p,\Delta_n})_L = V_m(\mathbb{C}^\flat_{p,\Delta_n})_L \subseteq \mathfrak{U}_{\delta,m-1,\mathbf{0}}.$$

Because $|\cdot|_{\text{prod}}$ equals 1 on nonzero elements of κ_L, it follows that

$$\mathfrak{U}_{\delta,m-1,\mathbf{0}} \cap W(\kappa_L)_L = V_m(\kappa_L)_L = \pi^m W(\kappa_L)_L.$$

By Lemma 1.46 for $R = W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ if follows that $\mathfrak{U}_{\delta,m-1,\mathbf{0}}/\pi^m W(\mathbb{C}_{p,\Delta_n}^\flat)_L$ is an open subset of $W_m(\mathbb{C}_{p,\Delta_n}^\flat)_L$. Clearly then

$$\mathfrak{U}_{\delta,m-1,\mathbf{0}}/\pi^m W(\mathbb{C}_{p,\Delta_n}^\flat)_L \bigcap (\mathcal{O}_L/\pi^m\mathcal{O}_L) = 0.$$

Therefore

$$\left(\mathfrak{U}_{\delta,m-1,\mathbf{0}}/\pi^m W(\mathbb{C}_{p,\Delta_n}^\flat)_L \cap \widehat{\mathcal{A}_{\Delta_n}^{\text{ur}}}/\pi^m \widehat{\mathcal{A}_{\Delta_n}^{\text{ur}}}\right) \bigcap (\mathcal{O}_L/\pi^m\mathcal{O}_L) = 0,$$

hence $U = \mathfrak{U}_{\delta,m-1,\mathbf{0}}/\pi^m W(\mathbb{C}_{p,\Delta_n}^\flat)_L \cap \widehat{\mathcal{A}_{\Delta_n}^{\text{ur}}}/\pi^m \widehat{\mathcal{A}_{\Delta_n}^{\text{ur}}}$ is an open set of $\widehat{\mathcal{A}_{\Delta_n}^{\text{ur}}}/\pi^m \widehat{\mathcal{A}_{\Delta_n}^{\text{ur}}}$ that satisfies the desired property. $\square$

3.6 The functors

In this section we introduce the functors that realize the desired equivalence of categories. Before we do that, we first compute the elements invariant under the Galois action and the Frobenius operators on $\widehat{\mathcal{A}_{\Delta_n}^{\text{ur}}}$. The strategy will be to compute these invariants on the characteristic p quotient ring $\widehat{\mathcal{A}_{\Delta_n}^{\text{ur}}}/\pi\widehat{\mathcal{A}_{\Delta_n}^{\text{ur}}} \simeq E_{\Delta_n}^{\text{sep}}$ and lift the results to the characteristic zero case. We also prove statements of greater generality that we will use in the later parts of the text.

3.6.1 Group invariants

For the desired group invariants, we begin with a general result that will help us compute the invariants of a group action on a tensor product.

Lemma 3.39. *Let R be a commutative ring and M be an R-module. Suppose that G is a group acting on M by R-linear automorphisms. Let N be a projective R-module. Consider the action of G on $M \otimes_R N$ given by the formula*

$$g(x \otimes y) = g(x) \otimes y$$

where $g \in G, x \in M$ and $y \in N$. Then

$$(M \otimes_R N)^G \simeq M^G \otimes_R N.$$

Proof. Using the projectivity of N, let P be an R-module such that $N \oplus P \simeq R^{\oplus S}$ for some set S. The isomorphisms

$$M \otimes_R (N \oplus P) \simeq M \otimes_R R^{\oplus S} \simeq M^{\oplus S}$$

are G-equivariant where we define the actions of G on $M \otimes_R (N \oplus P)$ and $M \otimes_R P$ in an analogous way to the one on $M \otimes_R N$. Taking G-invariants in the above isomorphisms we obtain

$$(M^G)^{\oplus S} = (M^{\oplus S})^G \simeq \big(M \otimes_R (N \oplus P)\big)^G \simeq (M \otimes_R N)^G \oplus (M \otimes_R P)^G.$$

Using the flatness of N and P over R we have that $M^G \otimes_R N$ and $M^G \otimes_R P$ are contained in $M \otimes_R N$ and $M \otimes_R P$, respectively. It is also clear that $M^G \otimes_R N \subseteq (M \otimes_R N)^G$ and $M^G \otimes_R P \subseteq (M \otimes_R P)^G$ hold. Hence

$$(M^G)^{\oplus S} = (M^G \otimes N) \oplus (M^G \otimes P) \subseteq (M \otimes_R N)^G \oplus (M \otimes_R P)^G = (M^G)^{\oplus S}$$

Therefore all of the inclusions must be equalities and the result follows. $\square$

We now proceed to compute the $H_{\Delta_n,L}$-invariants of $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$. As we explained above, we first compute these invariants in the quotient $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}/\pi\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \simeq E_{\Delta_n}^{\mathrm{sep}}$. By Proposition 1.40 and (3.27), we have natural group isomorphisms

$$\prod_{i\in\Delta_n} \mathrm{Gal}(\mathcal{B}_i^{\mathrm{ur}}/\mathcal{B}_i) \simeq \prod_{i\in\Delta_n} \mathrm{Gal}(E_i^{\mathrm{sep}}/E_i) \simeq H_{\Delta_n,L}. \tag{3.35}$$

Lemma 3.40. *Suppose that for all $i \in \Delta_n$, E_i' is a finite Galois extension of E_i and that $H_{i,L}' := \mathrm{Gal}(E_i^{\mathrm{sep}}/E_i')$. Then $(E_{\Delta_n}^{\mathrm{sep}})^{H_{\Delta_n,L}'} = E_{\Delta_n}'$ where $H_{\Delta_n,L}' := \prod_{i\in\Delta_n} H_{i,L}'$.*

Proof. It is clear that $(E_{\Delta_n}^{\mathrm{sep}})^{H_{\Delta_n,L}'} \supseteq E_{\Delta_n}'$. For the reverse inclusion, we check first the $H_{1,L}'$-invariants. We use the isomorphism from Lemma 3.10

$$E_{\Delta_n}^{\mathrm{sep}} \simeq E_1^{\mathrm{sep}} \otimes_{E_1} \left(\dots \left(E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n}\right)\right)$$

for which the formula of the Galois action from Lemma 3.22 becomes

$$\left(\prod_{i\in\Delta_n} \sigma_i\right) \Big(e_1 \otimes (\dots \otimes (e_n \otimes f))\Big) = \sigma_1(e_1) \otimes \left(\dots \otimes \left(\sigma_n(e_n) \otimes \left(\prod_{i\in\Delta_n} \overline{\sigma}_i\right)(f)\right)\right),$$

where $e_i \in E_i^{\mathrm{sep}}, f \in E_{\Delta_n}, \sigma_i \in G_{i,L}$ and $\overline{\sigma}_i = \sigma_i \bmod H_{i,L}$. The action of $H_{1,L}'$ on E_1^{sep} is E_1-linear while on $E_2^{\mathrm{sep}} \otimes_{E_2} \left(\dots \left(E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n}\right)\right)$ it is trivial, therefore by Lemma 3.39

$$\begin{aligned}
(E_{\Delta_n}^{\mathrm{sep}})^{H_{1,L}'} &\simeq (E_1^{\mathrm{sep}} \otimes_{E_1} (E_2^{\mathrm{sep}} \otimes_{E_2} \dots (E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n})))^{H_{1,L}'} \\
&\simeq (E_1^{\mathrm{sep}})^{H_{1,L}'} \otimes_{E_1} (E_2^{\mathrm{sep}} \otimes_{E_2} \dots (E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n})) \\
&\simeq E_1' \otimes_{E_1} (E_2^{\mathrm{sep}} \otimes_{E_2} \dots (E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n}))\,.
\end{aligned}$$

Thus $(E_{\Delta_n}^{\mathrm{sep}})^{H_{\Delta_n,L}'} \subseteq E_1' \otimes_{E_1} (E_2^{\mathrm{sep}} \otimes_{E_2} \dots (E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n}))$. Applying the same argument for the other components and actions of the $H_{i,L}'$ for $i \geq 2$ we obtain that

$$(E_{\Delta_n}^{\mathrm{sep}})^{H_{\Delta_n,L}'} \subseteq E_1' \otimes_{E_1} (E_2' \otimes_{E_2} \dots (E_n' \otimes_{E_n} E_{\Delta_n})) \simeq E_{\Delta_n}'$$

where the last isomorphism holds by Lemma 3.13 (iii). $\square$

Corollary 3.41. *(i)* $(E^{\mathrm{sep}}_{\Delta_n})^{H_{\Delta_n,L}} = E_{\Delta_n}$.

(ii) $(E^{\mathrm{sep}+}_{\Delta_n})^{H_{\Delta_n,L}} = E^{+}_{\Delta_n}$.

Proof. (i) This follows directly from Lemma 3.40 applied for the case when $E'_i = E_i$ for all $i \in \Delta_n$.

(ii) By Corollary 3.14 (ii) $E^{\mathrm{sep}+}_{\Delta_n}$ can be regarded as a submodule of $E^{\mathrm{sep}}_{\Delta_n}$. By the previous part it suffices to show that

$$E^{\mathrm{sep}+}_{\Delta_n} \cap E_{\Delta_n} = E^{+}_{\Delta_n}.$$

For this, it suffices to show that if we are given a collection $(E'_i/E_i)_{i\in\Delta_n}$ of finite separable extensions, then

$$E'^{+}_{\Delta_n} \cap E_{\Delta_n} = E^{+}_{\Delta_n}, \tag{3.36}$$

For $i \in \Delta_n$, E'^{+}_i and E'_i admit a common basis containing 1 over E^{+}_i and E_i, respectively. Therefore $E'^{+}_{\Delta_n}$ and E'_{Δ_n} admit a common basis containing 1 over $E^{+}_{\Delta_n}$ and E_{Δ_n}, respectively. This makes it clear that (3.36) holds and we are done. □

We also have the following cohomological result about the action of $H_{\Delta_n,L}$ on $E^{\mathrm{sep}}_{\Delta_n}$.

Lemma 3.42. *(i) $E^{\mathrm{sep}}_{\Delta_n}$ is a discrete $H_{\Delta_n,L}$-module, in other words, the map*

$$\begin{aligned} H_{\Delta_n,L} \times E^{\mathrm{sep}}_{\Delta_n} &\longrightarrow E^{\mathrm{sep}}_{\Delta_n} \\ (\sigma, e) &\longmapsto \sigma(e) \end{aligned}$$

is continuous for the discrete topology on $E^{\mathrm{sep}}_{\Delta_n}$.

(ii) $H^1_{\mathrm{cont}}(H_{\Delta_n,L}, E^{\mathrm{sep}}_{\Delta_n}) = 0$.

Proof. (i) It suffices to show that

$$E^{\mathrm{sep}}_{\Delta_n} = \bigcup_N (E^{\mathrm{sep}}_{\Delta_n})^N$$

where N runs through the open subgroups of $H_{\Delta_n,L}$. This is immediate from Corollary 3.41 as

$$E^{\mathrm{sep}}_{\Delta_n} \simeq \varinjlim_{\substack{[E'_i:E_i]<\infty \\ E'_i/E_i \text{ Galois}}} E'_{\Delta_n}.$$

(ii) $H_{\Delta_n,L}$ is a profinite group and $E^{\mathrm{sep}}_{\Delta_n}$ is a discrete $H_{\Delta_n,L}$-module, therefore by Corollary 1 of [Ser02]

$$H^1_{\mathrm{cont}}(H_{\Delta_n,L}, E^{\mathrm{sep}}_{\Delta_n}) = \varinjlim_N H^1(H_{\Delta_n,L}/N, (E^{\mathrm{sep}}_{\Delta_n})^N)$$

where N runs through the open normal subgroups of $H_{\Delta_n,L}$. By Lemma 3.40 it then suffices to show that

$$H^1\left(\prod_{i\in\Delta_n}\mathrm{Gal}(E_i'/E_i), E_{\Delta_n}'\right) = 0$$

where E_i' is a finite Galois extension of E_i for all $i \in \Delta_n$. Let $G := \prod_{i\in\Delta_n}\mathrm{Gal}(E_i'/E_i)$. By the normal basis theorem for each $i \in \Delta_n$ there exists a $b_i \in E_i'$ such that $\{\sigma(b_i)\}_{\sigma\in\mathrm{Gal}(E_i'/E_i)}$ is a basis of E_i' as an E_i-vector space. Then for

$$b := (b_1 \otimes \ldots \otimes b_n) \otimes 1 \in E_{\Delta_n}',$$

$\{\sigma(b)\}_{\sigma\in G}$ is a basis of E_{Δ_n}' over E_{Δ_n}. Let $c : G \longrightarrow E_{\Delta_n}'$ be a 1-cocycle, that is a map satisfying

$$c(\sigma\tau) = c(\sigma) + \sigma(c(\tau))$$

for all $\sigma, \tau \in G$. Write

$$c(\sigma) = \sum_{\tau\in G} c_\tau(\sigma)\tau(b)$$

for $\sigma \in G$ and $c_\tau(\sigma) \in E_{\Delta_n}$. Writing the cocycle condition, we compute

$$\begin{aligned}\sum_{\tau\in G} c_\tau(\sigma_1\sigma_2)\tau(b) = c(\sigma_1\sigma_2) &= c(\sigma_1) + \sigma_1(c(\sigma_2)) \\ &= \sum_{\tau\in G} c_\tau(\sigma_1)\tau(b) + \sum_{\tau\in G} c_\tau(\sigma_2)(\sigma_1\tau)(b) \\ &= \sum_{\tau\in G}(c_\tau(\sigma_1) + c_{\sigma_1^{-1}\tau}(\sigma_2))\tau(b)\end{aligned}$$

and see that the maps c_τ must satisfy $c_\tau(\sigma_1\sigma_2) = c_\tau(\sigma_1)+c_{\sigma_1^{-1}\tau}(\sigma_2)$ for all $\tau, \sigma_1, \sigma_2 \in G$. When $\tau = 1$ we obtain that

$$c_{\sigma_1^{-1}}(\sigma_2) = c_1(\sigma_1\sigma_2) - c_1(\sigma_1)$$

for any $\sigma_1, \sigma_2 \in G$. We now put

$$x := \sum_{\tau\in G} c_1(\tau^{-1})\tau(b) \in E_{\Delta_n}',$$

and we check that

$$\begin{aligned}\sigma(x) - x &= \sum_{\tau\in G} c_1(\tau^{-1})(\sigma\tau)(b) - \sum_{\tau\in G} c_1(\tau^{-1})\tau(b) \\ &= \sum_{\tau\in G}(c_1(\tau^{-1}\sigma) - c_1(\tau^{-1}))\tau(b) \\ &= \sum_{\tau\in G} c_\tau(\sigma)\tau(b) \\ &= c(\sigma)\end{aligned}$$

for any $\sigma \in G$. Therefore $c : G \longrightarrow E_{\Delta_n}'$ is a 1-coboundary, as desired. $\square$

Lemma 3.43. $(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}})^{H_{\Delta_n,L}} = \mathcal{A}_{\Delta_n}$.

Proof. By Corollary 3.41 (i) we have that

$$(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi\mathcal{A}^{\mathrm{ur}}_{\Delta_n})^{H_{\Delta_n,L}} = (E^{\mathrm{sep}}_{\Delta_n})^{H_{\Delta_n,L}} = E_{\Delta_n} \simeq \mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n}.$$

It is also clear that $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}^{H_{\Delta_n,L}} \supseteq \mathcal{A}_{\Delta_n}$. Therefore we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_{\Delta_n}/\pi^m\mathcal{A}_{\Delta_n} \arrow[r, "\pi"] \arrow[d] & \mathcal{A}_{\Delta_n}/\pi^{m+1}\mathcal{A}_{\Delta_n} \arrow[r, "\mathrm{pr}"] \arrow[d] & \mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{H_{\Delta_n,L}} \arrow[r, "\pi"] & \left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^{m+1}\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{H_{\Delta_n,L}} \arrow[r, "\mathrm{pr}"] & \left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{H_{\Delta_n,L}} &
\end{tikzcd}$$

whose rows are exact. By induction with respect to m we deduce that

$$\left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{H_{\Delta_n,L}} \simeq \mathcal{A}_{\Delta_n}/\pi^m\mathcal{A}_{\Delta_n}.$$

It follows that

$$\begin{aligned}
\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}^{H_{\Delta_n,L}} &= \left(\varprojlim_m \mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{H_{\Delta_n,L}} \\
&\simeq \varprojlim_m \left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{H_{\Delta_n,L}} \\
&\simeq \varprojlim_m \mathcal{A}_{\Delta_n}/\pi^m\mathcal{A}_{\Delta_n} \\
&\simeq \mathcal{A}_{\Delta_n}.
\end{aligned}$$

□

3.6.2 Frobenius invariants

We now compute the invariants for the Frobenius maps by starting again with a more general statement.

Lemma 3.44. *(i) The map* $\mathrm{id} - \varphi_n : E^{\mathrm{sep}}_{\Delta_n} \longrightarrow E^{\mathrm{sep}}_{\Delta_n}$ *is surjective,*

(ii) The natural map from $E^{\mathrm{sep}}_{\Delta_{n-1}}$ *into* $E^{\mathrm{sep}}_{\Delta_n}$ *is an embedding and its image equals the kernel of* $\mathrm{id} - \varphi_n : E^{\mathrm{sep}}_{\Delta_n} \longrightarrow E^{\mathrm{sep}}_{\Delta_n}$.

Proof. (i) The map $\mathrm{id} - \varphi_n$ is additive, therefore for surjectivity it suffices to check that a pure tensor

$$(e_1 \otimes \ldots \otimes e_n) \otimes e \in E^{\mathrm{sep}}_{\Delta_n}$$

is in the image of $\mathrm{id} - \varphi_n$, where $e_i \in E^{\mathrm{sep}}_i$ and $e \in E_{\Delta_n}$. Because $\mathrm{id} - \varphi_n$ is $E_{\Delta_{n-1}}$-linear, multiplying by a large enough power of $X_{\Delta_{n-1}}$, we may assume that

$e_j \in E_j^{\mathrm{sep}+}$ and that $\mathrm{val}_{X_j}(e) \geq 0$ for $j \in \Delta_{n-1}$. Choose an $r > 0$ large enough such that $X_n^r e_n \in E_n^{\mathrm{sep}+}$. Write

$$e = \sum_{j \geq k_0} f_j X_n^j$$

where $k_0 \in \mathbb{Z}_{\leq 0}$ and $f_j \in E_{\Delta_{n-1}}^+$. Then we have that

$$\begin{aligned}(e_1 \otimes \ldots \otimes e_n) \otimes e &= (e_1 \otimes \ldots \otimes e_n) \otimes \left(\sum_{j=k_0}^{r} f_j X_n^j \right) + (e_1 \otimes \ldots \otimes e_n) \otimes \left(\sum_{j \geq r+1} f_j X_n^j \right) \\ &= \sum_{j=k_0}^{r} (e_1 \otimes \ldots \otimes e_{n-1} \otimes X_n^j e_n) \otimes f_j + (e_1 \otimes \ldots \otimes e_n) \otimes \left(\sum_{j \geq r+1} f_j X_n^j \right).\end{aligned}$$

On E_n^{sep} the map $\mathrm{id} - \varphi_n$ is surjective because $X^q - X - b$ is a separable polynomial for all $b \in E_n^{\mathrm{sep}}$. Therefore for every $k_0 \leq j \leq r$, there exists an element $y_j \in E_n^{\mathrm{sep}}$ such that

$$(\mathrm{id} - \varphi_n)(y_j) = X_n^j e_n.$$

Then we have that

$$(\mathrm{id} - \varphi_n) \left(\sum_{j=k_0}^{r} (e_1 \otimes \ldots \otimes e_{n-1} \otimes y_j) \otimes f_j \right) = \sum_{j=k_0}^{r} (e_1 \otimes \ldots \otimes e_{n-1} \otimes X_n^j e_n) \otimes f_j.$$

Note that

$$(e_1 \otimes \ldots \otimes e_n) \otimes \left(\sum_{j \geq r+1} f_j X_n^j \right) = (e_1 \otimes \ldots \otimes e_{n-1} \otimes X_n^{r+1} e_n) \otimes \left(\sum_{j \geq r+1} f_j X_n^{j-r-1} \right).$$

Since $\mathrm{id} - \varphi_n$ is additive, it suffices to prove that

$$(e_1 \otimes \ldots \otimes e_n) \otimes e \in \mathrm{im}(\mathrm{id} - \varphi_n)$$

where $e_i \in E_i^{\mathrm{sep}+}$ for $i \in \Delta_n$, $e \in E_{\Delta_n}^+$ and $|e_n|_\flat < 1$. Write

$$e = \sum_{j \geq 0} g_j X_n^j$$

for $g_j \in E_{\Delta_{n-1}}^+$ and suppose that $e_i \in E_i'^+$, where $E_i'^+$ is the ring of integers of a finite extension E_i' of E_i for $i \in \Delta_n$. Because $E_{\Delta_n}'^+$ is X_n-adically complete, we have that

$$\begin{aligned}(e_1 \otimes \ldots \otimes e_n) \otimes e &= (e_1 \otimes \ldots \otimes e_n) \otimes \left(\sum_{j \geq 0} g_j X_n^j \right) \\ &= \sum_{j \geq 0} (e_1 \otimes \ldots \otimes e_n) \otimes g_j X_n^j \\ &= \sum_{j \geq 0} (e_1 \otimes \ldots \otimes e_{n-1} \otimes e_n X_n^j) \otimes g_j.\end{aligned}$$

By our assumption about e_n, we have that $X_n^j e_n \in E_n'^+$ and $|X_n^j e_n|_\flat < 1$ for every $j \geq 0$. For $b \in E_n'^+$ such that $|b|_\flat < 1$ we have that $\sum_{\ell \geq 0} b^{q^\ell} \in E_n'^+$ and

$$(\mathrm{id} - \varphi_n)\left(\sum_{\ell \geq 0} b^{q^\ell}\right) = \sum_{\ell \geq 0} b^{q^\ell} - \sum_{\ell \geq 0} b^{q^{\ell+1}} = \sum_{\ell \geq 0} b^{q^\ell} - \sum_{\ell \geq 1} b^{q^\ell} = b.$$

by the continuity of $\mathrm{id} - \varphi_n$. Then

$$y \coloneqq \sum_{j \geq 0} \left(e_1 \otimes \ldots \otimes e_{n-1} \otimes \sum_{\ell \geq 0} \left(e_n X_n^j\right)^{q^\ell}\right) \otimes g_j$$

is an element of $E_{\Delta_n}'^+$ since $E_{\Delta_n}'^+$ is X_n-adically complete and by the continuity of $\mathrm{id} - \varphi_n$ we have that

$$\begin{aligned}
(\mathrm{id} - \varphi_n)(y) &= (\mathrm{id} - \varphi_n)\left(\sum_{j \geq 0} \left(e_1 \otimes \ldots \otimes e_{n-1} \otimes \sum_{\ell \geq 0} \left(e_n X_n^j\right)^{q^\ell}\right) \otimes g_j\right) \\
&= \sum_{j \geq 0} \left(e_1 \otimes \ldots \otimes e_{n-1} \otimes (\mathrm{id} - \varphi_n)\left(\sum_{\ell \geq 0} \left(e_n X_n^j\right)^{q^\ell}\right)\right) \otimes g_j \\
&= \sum_{j \geq 0} (e_1 \otimes \ldots \otimes e_{n-1} \otimes e_n X_n^j) \otimes g_j \\
&= (e_1 \otimes \ldots \otimes e_n) \otimes e
\end{aligned}$$

and we are done.

(ii) Using that E_{Δ_n} is flat over E_n, the inclusion $E_n \to E_n^{\mathrm{sep}}$ induces an embedding

$$E_{\Delta_n} \simeq E_n \otimes_{E_n} E_{\Delta_n} \to E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n}. \tag{3.37}$$

After precomposing (3.37) with the inclusion $E_{\Delta_{n-1}} \to E_{\Delta_n}$, we obtain that the map

$$\begin{aligned}
E_{\Delta_{n-1}} &\to E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n} \\
f &\mapsto 1 \otimes f
\end{aligned}$$

is an embedding. Using that E_i^{sep} is flat over E_i for $i \in \Delta_{n-1}$, we obtain that the map

$$\begin{aligned}
E_1^{\mathrm{sep}} \otimes_{E_1} \left(\ldots \left(E_{n-1}^{\mathrm{sep}} \otimes_{E_{n-1}} E_{\Delta_{n-1}}\right)\right) &\to E_1^{\mathrm{sep}} \otimes_{E_1} \left(\ldots \left(E_{n-1}^{\mathrm{sep}} \otimes_{E_{n-1}} (E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_n})\right)\right) \\
e_1 \otimes (\ldots \otimes (e_{n-1} \otimes f)) &\mapsto e_1 \otimes (\ldots \otimes (e_{n-1} \otimes (1 \otimes f)))
\end{aligned}$$

is an embedding, therefore by Lemma 3.10 (i) the natural map from $E_{\Delta_{n-1}}^{\mathrm{sep}}$ into $E_{\Delta_n}^{\mathrm{sep}}$ is an embedding.

For the statement about the kernel, note that the map $\mathrm{id} - \varphi_n$ is $E_{\Delta_{n-1}}^{\mathrm{sep}}$-linear and that $1 \in \ker(\mathrm{id} - \varphi_n)$, therefore it is clear that $E_{\Delta_{n-1}}^{\mathrm{sep}} \subseteq \ker(\mathrm{id} - \varphi_n)$. For the reverse inclusion, let $(E'_i/E_i)_{i\in\Delta_n}$ be a collection of finite separable extensions and $y \in E'_{\Delta_n}$ be an element of $\ker(\mathrm{id} - \varphi_n)$. Writing $E'_i = \mathbb{F}_{q_i}(\!(X'_i)\!)$ for $i \in \Delta_n$, by Corollary 3.16 (ii) and the fact that $X_i \in (X'_i)^{s_i}(E'^{+}_i)^\times$ for some $s_i \in \mathbb{N}$, we know that

$$E'_{\Delta_n} \simeq \left(\bigotimes_{i\in\Delta_n,\kappa_L} \mathbb{F}_{q_i}\right)[\![X'_1,\dots,X'_n]\!][(X'_1\dots X'_n)^{-1}].$$

Under this identification φ_n maps the monomial X'_i to X'_i for $i \in \Delta_{n-1}$, X'_n to X'^q_n and the pure tensors $c_1 \otimes \dots \otimes c_n$ in $\left(\bigotimes_{i\in\Delta_n,\kappa_L} \mathbb{F}_{q_i}\right)$ to $c_1 \otimes \dots \otimes c_{n-1} \otimes c_n^q$. Write

$$y = \sum_{j\in\mathbb{Z}} f_j X'^j_n$$

for $f_j \in \left(\bigotimes_{i\in\Delta_n,\kappa_L} \mathbb{F}_{q_i}\right)[\![X'_1,\dots,X'_{n-1}]\!][(X'_1\dots X'_{n-1})^{-1}]$. The map φ_n is injective on $\mathbb{F}_{q_n}$, therefore by flatness, it is also injective on $\bigotimes_{i\in\Delta_n,\kappa_L} \mathbb{F}_{q_i}$. Comparing the terms with the smallest value of $\mathrm{val}_{X'_n}$ in the equality $\varphi_n(y) = y$, it follows that $f_j = 0$ for all $j < 0$. Comparing the terms for which $\mathrm{val}_{X'_n} = 0$ in the same equality, it follows that $\varphi_n(f_0) = f_0$. We also have that

$$\ker\left(\mathrm{id} - \varphi_n : \bigotimes_{i\in\Delta_n,\kappa_L} \mathbb{F}_{q_i} \longrightarrow \bigotimes_{i\in\Delta_n,\kappa_L} \mathbb{F}_{q_i}\right) \simeq \bigotimes_{i\in\Delta_{n-1},\kappa_L} \mathbb{F}_{q_i}. \tag{3.38}$$

Indeed, let z be an element in the left hand side of (3.38) and suppose that $d_1,\dots,d_s$ is a κ_L-basis of $\bigotimes_{i\in\Delta_{n-1},\kappa_L} \mathbb{F}_{q_i}$. Write $z = \sum_{j=1}^{s} d_j \otimes b_j$ for some $b_j \in \mathbb{F}_{q_n}$. Then

$$\sum_{j=1}^{s} d_j \otimes b_j = z = \varphi_n(z) = \sum_{j=1}^{s} d_j \otimes b_j^q.$$

This implies that $b_j^q = b_j$, hence $b_j \in \kappa_L$ for every $1 \le j \le s$, which gives the claim of (3.38). From this it also follows that $f_0 \in E'_{\Delta_{n-1}}$. Because $(\mathrm{id} - \varphi_n)(f_0) = 0$, the additivity of $\mathrm{id} - \varphi_n$ implies that

$$\sum_{j>0} f_j X'^j_n \in \ker(\mathrm{id} - \varphi_n).$$

Checking the term with the smallest value of $\mathrm{val}_{X'_n}$ in the equality

$$\sum_{j>0} f_j X'^j_n = \varphi_n\left(\sum_{j>0} f_j X'^j_n\right),$$

we can inductively show that $f_j = 0$ for all $j > 0$. Thus $y \in E'_{\Delta_{n-1}} \subseteq E^{\mathrm{sep}}_{\Delta_{n-1}}$, as desired. $\square$

An immediate corollary to Lemma 3.44 shows that the elements of $E^{\mathrm{sep}}_{\Delta_n}$ simultaneously fixed by the operators φ_i form the field κ_L.

Corollary 3.45. $\displaystyle\bigcap_{i\in\Delta_n}\left(E^{\mathrm{sep}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}}=\kappa_L.$

Proof. Applying succesively Lemma 3.44 we obtain that $\displaystyle\bigcap_{i\in\{2,\ldots,n\}}\left(E^{\mathrm{sep}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}}=E^{\mathrm{sep}}_1.$ It is clear that $(E^{\mathrm{sep}}_1)^{\varphi_1=\mathrm{id}}=\kappa_L$. □

Lemma 3.46. $\displaystyle\bigcap_{i\in\Delta_n}(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}})^{\varphi_i=\mathrm{id}}=\mathcal{O}_L.$

Proof. By Corollary 3.45

$$\bigcap_{i\in\Delta_n}\left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}}\simeq\bigcap_{i\in\Delta_n}\left(E^{\mathrm{sep}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}}=\kappa_L\simeq\mathcal{O}_L/\pi\mathcal{O}_L.$$

Also, it is clear that $\displaystyle\bigcap_{i\in\Delta_n}\left(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}\right)^{\varphi_i=\mathrm{id}}\supseteq\mathcal{O}_L$. Therefore we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{O}_L/\pi^m\mathcal{O}_L \arrow[r] \arrow[d, "\pi"] & \bigcap_{i\in\Delta_n}\left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}} \arrow[d, "\pi"] \\
\mathcal{O}_L/\pi^{m+1}\mathcal{O}_L \arrow[r] \arrow[d, "\mathrm{pr}"] & \bigcap_{i\in\Delta_n}\left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^{m+1}\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}} \arrow[d, "\mathrm{pr}"] \\
\mathcal{O}_L/\pi\mathcal{O}_L \arrow[r] \arrow[d] & \bigcap_{i\in\Delta_n}\left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}} \\
0 &
\end{tikzcd}$$

whose columns are exact. By induction with respect to m we deduce that

$$\bigcap_{i\in\Delta_n}\left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}}=\mathcal{O}_L/\pi^m\mathcal{O}_L.$$

Because the φ_i operators commute with the projections modulo π^m, it follows that

$$\bigcap_{i\in\Delta_n}(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}})^{\varphi_i=\mathrm{id}}=\bigcap_{i\in\Delta_n}\left(\varprojlim_m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}}\simeq\varprojlim_m\bigcap_{i\in\Delta_n}\left(\mathcal{A}^{\mathrm{ur}}_{\Delta_n}/\pi^m\mathcal{A}^{\mathrm{ur}}_{\Delta_n}\right)^{\varphi_i=\mathrm{id}}$$
$$\simeq\varprojlim_m\mathcal{O}_L/\pi^m\mathcal{O}_L\simeq\mathcal{O}_L.$$

$\square$

3.6.3 The functors

Let $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. Consider the $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-module $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T$ which carries a diagonal $G_{\Delta_n,L}$-action and the Frobenius endomorphisms $\varphi_i \coloneqq \varphi_i \otimes \mathrm{id}$. These operators and the group action commute, hence by Lemma 3.43

$$\mathbb{D}(T) \coloneqq \left(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T\right)^{H_{\Delta_n,L}}$$

is an $\mathcal{A}_{\Delta_n}$-module with a semilinear action of $\mathcal{T}_{+,\Delta_n,L}$. Therefore

$$\mathbb{D}(T) \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$$

provided that we show it is a finitely generated $\mathcal{A}_{\Delta_n}$-module and that it is étale.

For morphisms, let $T_1, T_2 \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ and $f \in \mathrm{Mor}_{\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})}(T_1, T_2)$. Then the $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-linear map

$$\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_1 \xrightarrow{\mathrm{id}\otimes f} \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_2$$

respects the φ_i operators and is $G_{\Delta_n,L}$-equivariant for the diagonal action, therefore it restricts to an $\mathcal{A}_{\Delta_n}$-linear map

$$\mathbb{D}(f) : \mathbb{D}(T_1) \to \mathbb{D}(T_2)$$

that commutes with the action of $\mathcal{T}_{+,\Delta_n,L}$ on both sides.

Let $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. The $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-module $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D$ carries a diagonal $G_{\Delta_n,L}$-action where $G_{\Delta_n,L}$ acts on D through its quotient $\Gamma_{\Delta_n,L}$, as well as Frobenius operators

$$\begin{aligned}\varphi_i : \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D &\longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D \\ a \otimes d &\longmapsto \varphi_i(a) \otimes \varphi_i(d).\end{aligned}$$

The Frobenius operators and the $G_{\Delta_n,L}$-action commute, hence there is a $G_{\Delta_n,L}$-action on

$$\mathbb{T}(D) \coloneqq \bigcap_{i\in\Delta_n} \left(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D\right)^{\varphi_i=\mathrm{id}}.$$

By Lemma 3.46, $\mathbb{T}(D)$ is an $\mathcal{O}_L$-submodule of $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D$. Then $\mathbb{T}(D) \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ provided we show it is finitely generated over $\mathcal{O}_L$ and that $G_{\Delta_n,L}$ acts continuously on it with respect to the π-adic topology.

For morphisms, let $D_1, D_2 \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ and $g \in \mathrm{Mor}_{\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n},\Gamma_{\Delta_n,L},\mathcal{A}_{\Delta_n})}(D_1, D_2)$. Then the $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-linear map

$$\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_1 \xrightarrow{\mathrm{id}\otimes g} \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_2$$

respects the φ_i operators and is $G_{\Delta_n,L}$-equivariant for the diagonal action, therefore it restricts to an $\mathcal{O}_L$-linear $G_{\Delta_n,L}$-equivariant map

$$\mathbb{T}(g) : \mathbb{T}(D_1) \to \mathbb{T}(D_2).$$

In the next chapters we will show that $\mathbb{D}$ and $\mathbb{T}$ give quasi-inverse functors between $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ which realize the equivalence between these two categories.

Chapter 4

Equivalence for mod π coefficients

In this chapter, we focus our attention on the objects of our categories that are annihilated by π. On the one hand, in Remark 2.29 we already introduced the category $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ of étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over E_{Δ_n}. On the other hand, let $\mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$ denote the category of continuous representations with coefficients in $\mathcal{O}_L/\pi\mathcal{O}_L \simeq \kappa_L$. The π-adic topology of a finitely generated κ_L-vector space viewed as an $\mathcal{O}_L$-module is the discrete topology, therefore a finite dimensional linear representation of $G_{\Delta_n,L}$ with coefficients in κ_L is continuous if and only if there exists an open subgroup of $G_{\Delta_n,L}$ that fixes every element of the underlying vector space.

For the purposes of this chapter, for $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ we will instead denote $\mathbb{T}(D)$ by $\mathbb{V}(D)$ and will also write $\mathbb{V}(f)$ instead of $\mathbb{T}(f)$ if f is a morphism of étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over E_{Δ_n}. Our goal of the chapter is to show that the categories $\mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ are equivalent and that $\mathbb{D}$ and $\mathbb{V}$ are quasi-inverse functors that realize their equivalence. We follow entirely the exposition and proofs of Chapters 2 and 3 of [Záb18b], although we will restructure and provide a more detailed account of some of its arguments here. For this we added some new lemmas to explain some of the intermediary steps.

4.1 Reduction to the essential surjectivity of $\mathbb{D}$

The first part of the desired equivalence is realized by the following proposition, whose proof is entirely analogous to that of Lemma 3.4 in [Záb18b]. It begins with the observation that if $V \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$, then by restriction V is a finite-dimensional continuous representation of $H_{\Delta_n,L}$ with coefficients in κ_L, which we shortly write as $V \in \mathrm{Rep}_{\kappa_L}(H_{\Delta_n,L})$.

Proposition 4.1. *The space $E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} V$ has an $E_{\Delta_n}^{\mathrm{sep}}$-basis of elements fixed by $H_{\Delta_n,L}$.*

Proof. Since $E^{\text{sep}}_{\Delta_n} \otimes_{\kappa_L} V \simeq E^{\text{sep}}_{\Delta_n} \otimes_{E^{\text{sep}}_{\Delta_n,\circ}} \left(E^{\text{sep}}_{\Delta_n,\circ} \otimes_{\kappa_L} V\right)$ and this identification is $H_{\Delta_n,L}$-equivariant it suffices to show that $E^{\text{sep}}_{\Delta_n,\circ} \otimes_{\kappa_L} V$ has an $E^{\text{sep}}_{\Delta_n,\circ}$-basis fixed by $H_{\Delta_n,L}$.

We will reduce the claim to the case when $n = 1$ which is proven in Proposition 3.2.1 of [Sch17]. Suppose that $n \geq 2$ and note that V is also a continuous representation with coefficients in κ_L of $G_{n,L}$. By the result in the one variable case, we can consider an E^{sep}_n-basis $e_1, \ldots, e_d$ of $E^{\text{sep}}_n \otimes_{\kappa_L} V$ fixed by $H_{n,L}$, where $d = \dim_{\kappa_L} V$.

Let $V_n := E_n e_1 \oplus \ldots \oplus E_n e_d = (E^{\text{sep}}_n \otimes_{\kappa_L} V)^{H_{n,L}}$. Then V_n is a linear representation of $H_{\Delta_{n-1},L}$ with coefficients in E_n. Also

$$\begin{aligned} E^{\text{sep}}_n \otimes_{E_n} V_n &\longrightarrow E^{\text{sep}}_n \otimes_{\kappa_L} V \\ e \otimes v &\longmapsto ev \end{aligned} \tag{4.1}$$

is an $H_{\Delta_{n-1},L}$-equivariant E^{sep}_n-linear isomorphism. Let $\{v_1, \ldots, v_d\}$ be a κ_L-basis of V. Let $B \in \mathrm{GL}_d(E^{\text{sep}}_n)$ be the matrix of the above isomorphism with respect to the bases $\{1 \otimes e_1, \ldots, 1 \otimes e_d\}$ and $\{1 \otimes v_1, \ldots, 1 \otimes v_d\}$, respectively. Since B has finitely many separable algebraic entries over E_n, it follows that $B \in \mathrm{GL}_d(E'_n)$, where $E_n \subseteq E'_n$ is a finite separable extension. By the primitive element theorem, consider $u \in E'_n$ with $E'_n = E_n(u)$ and write

$$B = B_0 + B_1 u + \ldots + B_m u^m$$

for $B_i \in \mathrm{M}_d(E_n)$. For $h \in H_{\Delta_{n-1},L}$, let $\rho(h) \in \mathrm{GL}_d(\kappa_L)$ be the matrix of the action of h on V with respect to the basis $\{v_1, \ldots, v_d\}$ and $\rho_n(h) \in \mathrm{GL}_d(E_n)$ be the matrix of the action of h on V_n, with respect to the basis $\{e_1, \ldots, e_d\}$. Then $B\rho_n(h) = \rho(h)B$ by (4.1) for every $h \in H_{\Delta_{n-1},L}$. Thus

$$B_i \rho_n(h) = \rho(h) B_i \tag{4.2}$$

for all $1 \leq i \leq m$ and $h \in H_{\Delta_{n-1},L}$. Consider the polynomial

$$f(t) = \det(B_0 + B_1 t + \ldots + B_m t^m) \in E_n[t].$$

Since $f(u) \neq 0$, then f is not the zero polynomial in $E_n[t]$. As E_n is a field with infinitely many elements, there exists an element $u_0 \in E_n$ such that $f(u_0) \neq 0$. By (4.2), we have

$$(B_0 + \ldots + B_m u_0^m)\rho_n(h) = \rho(h)(B_0 + \ldots + B_m u_0^m)$$

for every $h \in H_{\Delta_{n-1},L}$. As $f(u_0) = B_0 + \ldots + B_m u_0^m \in \mathrm{GL}_d(E_n)$, it follows that $E_n \otimes_{E_n} V_n \simeq V_n$ and $E_n \otimes_{\kappa_L} V$ are isomorphic representations of $H_{\Delta_{n-1},L}$ with coefficients in E_n. Therefore, V_n admits a basis $\widetilde{e}_1, \ldots, \widetilde{e}_d$ with respect to which the actions of each $h \in H_{\Delta_{n-1},L}$ are given by matrices in $\mathrm{GL}_d(\kappa_L)$.

Let $W_n = \kappa_L \widetilde{e}_1 \oplus \ldots \oplus \kappa_L \widetilde{e}_d$. Then W_n is a linear representation of $H_{\Delta_{n-1},L}$ with coefficients in κ_L. Moreover, we claim it is a continuous representation, meaning that $W_n \in \operatorname{Rep}_{\kappa_L}(H_{\Delta_{n-1},L})$. For that we need to show that each element of W_n is fixed by an open subgroup of $H_{\Delta_{n-1},L}$. We know that $W_n \subseteq V_n \subseteq E_n^{\text{sep}} \otimes_{\kappa_L} V$, thus we can write any element of W_n as an E_n^{sep}-linear combination of $\{1 \otimes v_1, \dots, 1 \otimes v_d\}$. Since $V \in \operatorname{Rep}_{\kappa_L}(H_{\Delta_n,L})$, we know that there exists an open subgroup of $H_{\Delta_{n-1},L}$ fixing each of the v_i and as $H_{\Delta_{n-1},L}$ acts trivially on E_n^{sep} the conclusion follows.

Suppose that $E_{\Delta_{n-1},\circ}^{\text{sep}} \otimes_{\kappa_L} W_n$ has an $E_{\Delta_{n-1},\circ}^{\text{sep}}$-basis $w_1, \dots, w_d$ fixed by $H_{\Delta_{n-1},L}$. We also have that

$$\begin{aligned} E_{\Delta_{n-1},\circ}^{\text{sep}} \otimes_{\kappa_L} W_n &\subseteq E_{\Delta_{n-1},\circ}^{\text{sep}} \otimes_{\kappa_L} (E_n^{\text{sep}} \otimes_{\kappa_L} V)^{H_{n,L}} \\ &= \left(E_{\Delta_n,\circ}^{\text{sep}} \otimes_{\kappa_L} V\right)^{H_{n,L}} \end{aligned}$$

where the last equality follows from Lemma 3.39. This means that $\{w_1, \dots, w_d\} \subseteq \left(E_{\Delta_n,\circ}^{\text{sep}} \otimes_{\kappa_L} V\right)^{H_{\Delta_n,L}}$. Since

$$\begin{aligned} E_{\Delta_{n-1},\circ}^{\text{sep}} \otimes_{\kappa_L} (E_n^{\text{sep}} \otimes_{E_n} (E_n \otimes_{\kappa_L} W_n)) &\simeq E_{\Delta_{n-1},\circ}^{\text{sep}} \otimes_{\kappa_L} (E_n^{\text{sep}} \otimes_{E_n} V_n) \\ &\simeq E_{\Delta_{n-1},\circ}^{\text{sep}} \otimes_{\kappa_L} (E_n^{\text{sep}} \otimes_{\kappa_L} V) \\ &\simeq E_{\Delta_n,\circ}^{\text{sep}} \otimes_{\kappa_L} V \end{aligned}$$

it follows that $w_1, \dots, w_d$ can be identified with a basis of $E_{\Delta_n,\circ}^{\text{sep}} \otimes_{\kappa_L} V$ and is fixed by $H_{\Delta_n,L}$. Therefore, we can reduce the value of n until it equals 1 to get the desired conclusion. $\square$

The space $E_{\Delta_n}^{\text{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V)$ comes equipped with a diagonal $G_{\Delta_n,L}$-action, where $G_{\Delta_n,L}$ acts on $\mathbb{D}(V)$ through its quotient $\Gamma_{\Delta_n,L}$. It also comes equipped with φ_i operators that act diagonally using the existings φ_i operators on $E_{\Delta_n}^{\text{sep}}$ and $\mathbb{D}(V)$.

Corollary 4.2. *Let $V \in \operatorname{Rep}_{\kappa_L}(G_{\Delta_n,L})$. Then*

(i) $\mathbb{D}(V)$ is a free module over E_{Δ_n} of finite rank equal to $\dim_{\kappa_L} V$.

(ii) The module $\mathbb{D}(V)$ is étale, hence $\mathbb{D}$ is a well-defined functor

$$\mathbb{D} : \operatorname{Rep}_{\kappa_L}(G_{\Delta_n,L}) \to \operatorname{Mod}^{ét}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n}).$$

(iii) The natural map

$$\begin{aligned} \operatorname{ad}_V : E_{\Delta_n}^{\text{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V) &\to E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V \\ e \otimes w &\mapsto ew \end{aligned}$$

is an isomorphism respecting the actions of $G_{\Delta_n,L}$ and of the operators φ_i on both sides.

Proof. (i) Using the notation of the proof of Proposition 4.1 we know that

$$E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V = E_{\Delta_n}^{\text{sep}} w_1 \oplus \ldots \oplus E_{\Delta_n}^{\text{sep}} w_d$$

where $d = \dim_{\kappa_L} V$ and $\{w_1, \ldots, w_d\}$ is our $E_{\Delta_n}^{\text{sep}}$-basis of $E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V$ fixed by $H_{\Delta_n,L}$. Thus, by Corollary 3.41 (i)

$$\begin{aligned}\mathbb{D}(V) = \left(E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V\right)^{H_{\Delta_n,L}} &= \left(E_{\Delta_n}^{\text{sep}}\right)^{H_{\Delta_n,L}} w_1 \oplus \ldots \oplus \left(E_{\Delta_n}^{\text{sep}}\right)^{H_{\Delta_n,L}} w_d \\ &= E_{\Delta_n} w_1 \oplus \ldots \oplus E_{\Delta_n} w_d.\end{aligned}$$

(ii) Consider again our E_{Δ_n}-basis $\{w_1, \ldots, w_d\}$ of $\mathbb{D}(V)$ from part (i) and let $i \in \Delta_n$ be arbitrary. Since $\mathbb{D}(V)$ is free by part (i), we have that

$$\left[\varphi_i(w_1) \ldots \varphi_i(w_d)\right]^t = A_i \left[w_1 \ldots w_d\right]^t$$

for a unique $A_i \in \mathrm{M}_d(E_{\Delta_n})$. Then A_i is also the matrix of the linearized map

$$\begin{aligned}\varphi_i^{\text{lin}} : E_{\Delta_n} \otimes_{\varphi_i, E_{\Delta_n}} \mathbb{D}(V) &\longrightarrow \mathbb{D}(V) \\ e \otimes x &\longmapsto e\varphi_i(x)\end{aligned}$$

with respect to the bases $\{1 \otimes w_1, \ldots, 1 \otimes w_d\}$ and $\{w_1, \ldots, w_d\}$, respectively. We need to show that $\det(A_i) \in E_{\Delta_n}^{\times}$. Let $v_1, \ldots, v_d$ be a basis of V and write

$$\left[w_1 \ldots w_d\right]^t = C \left[1 \otimes v_1 \ldots 1 \otimes v_d\right]^t$$

where $C = (c_{jk})_{1 \le j,k \le d} \in \mathrm{M}_d(E_{\Delta_n}^{\text{sep}})$. Since both $\{1 \otimes v_1, \ldots, 1 \otimes v_d\}$ and $\{w_1, \ldots, w_d\}$ are bases of $E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V$, it follows that $C \in \mathrm{GL}_d(E_{\Delta_n}^{\text{sep}})$. Computing

$$\begin{aligned}\left[\varphi_i(w_1) \ldots \varphi_i(w_d)\right]^t &= \left[\varphi_i\left(\sum_{j=1}^d c_{1j} \otimes v_j\right) \ldots \varphi_i\left(\sum_{j=1}^d c_{dj} \otimes v_j\right)\right]^t \\ &= \left[\sum_{j=1}^d \varphi_i(c_{1j}) \otimes v_j \ldots \sum_{j=1}^d \varphi_i(c_{dj}) \otimes v_j\right]^t \\ &= \varphi_i(C) \left[1 \otimes v_1 \ldots 1 \otimes v_d\right]^t \\ &= \varphi_i(C) C^{-1} \left[w_1 \ldots w_d\right]^t,\end{aligned}$$

it follows that $A_i = \varphi_i(C)C^{-1}$ and $\det(A_i) = \varphi_i(\det(C)) \det(C^{-1})$. Since $\det(C) \in E_{\Delta_n}^{\text{sep}\,\times}$ and φ_i is a ring endomorphism of $E_{\Delta_n}^{\text{sep}}$, φ_i preserves units and therefore

$$\det(A_i) \in E_{\Delta_n}^{\text{sep}\,\times} \cap E_{\Delta_n}.$$

By Lemma 3.41 (i), we have that $\det(A_i)$ is a unit in $E_{\Delta_n}^{\text{sep}}$ fixed by every $\tau \in H_{\Delta_n,L}$. Since every such τ is an automorphism of $E_{\Delta_n}^{\text{sep}}$, then

$$\tau\left(\frac{1}{\det(A_i)}\right) = \frac{1}{\tau(\det(A_i))} = \frac{1}{\det(A_i)}.$$

Applying Lemma 3.41 (i), it follows that $\dfrac{1}{\det(A_i)} \in E_{\Delta_n}$. This implies that $\det(A_i) \in E_{\Delta_n}^{\times}$ and that $\mathbb{D}(V)$ is étale.

(iii) The fact that $\mathbb{D}(V) = E_{\Delta_n} w_1 \oplus \ldots \oplus E_{\Delta_n} w_d$ by part (i), makes it clear that our proposed map

$$\begin{aligned} E_{\Delta_n}^{\text{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V) &\xrightarrow{\text{ad}_V} E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V \\ e \otimes w &\longmapsto ew \end{aligned}$$

is an isomorphism of $E_{\Delta_n}^{\text{sep}}$-modules. Checking that ad_V respects the Galois action and the Frobenius operators is done directly by the equalities

$$\begin{aligned} \text{ad}_V(\sigma(e \otimes w)) &= \text{ad}_V(\sigma(e) \otimes \sigma(w)) = \sigma(e)\sigma(w) \\ &= \sigma(ew) = \sigma(\text{ad}_V(e \otimes w)) \end{aligned}$$

and

$$\begin{aligned} \text{ad}_V(\varphi_i(e \otimes w)) &= \text{ad}_V(\varphi_i(e) \otimes \varphi_i(w)) = \varphi_i(e)\varphi_i(w) \\ &= \varphi_i(ew) = \varphi_i(\text{ad}_V(e \otimes w)) \end{aligned}$$

which hold due to the semilinearity of our actions on $E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V$, for every $e \in E_{\Delta_n}^{\text{sep}}, w \in \mathbb{D}(V), \sigma \in G_{\Delta_n,L}$ and $i \in \Delta_n$. □

By Corollary 4.2 (iii), the map ad_V restricts to an isomorphism

$$\text{ad}_V : \bigcap_{i \in \Delta_n} \left(E_{\Delta_n}^{\text{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V)\right)^{\varphi_i} \to \bigcap_{i \in \Delta_n} \left(E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V\right)^{\varphi_i = \text{id}},$$

therefore for every $V \in \text{Rep}_{\kappa_L}(G_{\Delta_n,L})$ we can consider the composition of maps

$$\text{adj}_V : V \xrightarrow{\eta_V} \bigcap_{i \in \Delta_n} \left(E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V\right)^{\varphi_i = \text{id}} \xrightarrow{\text{ad}_V^{-1}} \bigcap_{i \in \Delta_n} \left(E_{\Delta_n}^{\text{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V)\right)^{\varphi_i = \text{id}} = \mathbb{V}(\mathbb{D}(V))$$

where $\eta_V(v) \coloneqq 1 \otimes v$ for $v \in V$.

Corollary 4.3. *For $V \in \text{Rep}_{\kappa_L}(G_{\Delta_n,L})$ we have that $\mathbb{V}(\mathbb{D}(V)) \in \text{Rep}_{\kappa_L}(G_{\Delta_n,L})$ and the maps $\{\text{adj}_V\}_{V \in \text{Rep}_{\kappa_L}(G_{\Delta_n,L})}$ give a natural isomorphism of functors*

$$\text{id}_{\text{Rep}_{\kappa_L}(G_{\Delta_n,L})} \simeq \mathbb{V} \circ \mathbb{D}.$$

Proof. The operators φ_i act trivially on V, therefore any κ_L-basis of V extends to an $E_{\Delta_n}^{\text{sep}}$-basis of $E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V$ fixed by each φ_i. Therefore the map

$$\eta_V : V \to \bigcap_{i \in \Delta_n} \left(E_{\Delta_n}^{\text{sep}} \otimes_{\kappa_L} V\right)^{\varphi_i = \text{id}}$$

is an isomorphism by Corollary 3.45. By Corollary 4.2 (iii) it follows that the map

$$\text{adj}_V : V \to \mathbb{V}(\mathbb{D}(V))$$

is a $G_{\Delta_n,L}$-equivariant isomorphism as well. This shows that $\mathbb{V}(\mathbb{D}(V))$ is a finite dimensional κ_L-vector space. The continuity of V and the equivariance of adj_V show that there exists an open subgroup of $G_{\Delta_n,L}$ that fixes every element of $\mathbb{V}(\mathbb{D}(V))$, therefore

$$\mathbb{V}(\mathbb{D}(V)) \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L}).$$

We are left to show that for $V_1, V_2 \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$ and $f \in \mathrm{Mor}_{\mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})}(V_1, V_2)$, the diagram

$$\begin{tikzcd}
V_1 \arrow[r, "f"] \arrow[d, "\mathrm{adj}_{V_1}"] & V_2 \arrow[d, "\mathrm{adj}_{V_2}"] \\
\mathbb{V}(\mathbb{D}(V_1)) \arrow[r, "\mathbb{V}(\mathbb{D}(f))"] & \mathbb{V}(\mathbb{D}(V_2))
\end{tikzcd}$$

commutes. We break our diagram into

$$\begin{tikzcd}
V_1 \arrow[r, "f"] \arrow[d, "\eta_{V_1}"] & V_2 \arrow[d, "\eta_{V_2}"] \\
\bigcap_{i\in\Delta_n} \left(E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} V_1\right)^{\varphi_i=\mathrm{id}} \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "\mathrm{ad}_{V_1}^{-1}"] & \bigcap_{i\in\Delta_n} \left(E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} V_2\right)^{\varphi_i=\mathrm{id}} \arrow[d, "\mathrm{ad}_{V_2}^{-1}"] \\
\mathbb{V}(\mathbb{D}(V_1)) \arrow[r, "\mathbb{V}(\mathbb{D}(f))"] & \mathbb{V}(\mathbb{D}(V_2))
\end{tikzcd} \tag{4.3}$$

and show that both squares commute. For the upper one, we compute directly that

$$(\mathrm{id}\otimes f)\left(\eta_{V_1}(v)\right) = (\mathrm{id}\otimes f)(1\otimes v) = 1 \otimes f(v) = \eta_{V_2}(f(v))$$

holds for every $v \in V_1$. For the lower square, we first show that the diagram

$$\begin{tikzcd}
E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} V_1 \arrow[r, "\mathrm{id}\otimes f"] & E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} V_2 \\
E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V_1) \arrow[r, "\mathrm{id}\otimes\mathbb{D}(f)"] \arrow[u, "\mathrm{ad}_{V_1}"] & E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V_2) \arrow[u, "\mathrm{ad}_{V_2}"]
\end{tikzcd} \tag{4.4}$$

commutes. To check this, let

$$e \otimes \left(\sum_{i=1}^{r} e_i \otimes v_i\right) \in E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V_1)$$

for $e, e_i \in E_{\Delta_n}^{\mathrm{sep}}$ and $v_i \in V_1$ when $1 \le i \le r$ such that

$$\sum_{i=1}^{r} e_i \otimes v_i \in \mathbb{D}(V_1) \subseteq E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} V_1.$$

By direct computation

$$\begin{aligned}
\mathrm{ad}_{V_2} \circ (\mathrm{id} \otimes \mathbb{D}(f)) \left(e \otimes \left(\sum_{i=1}^{r} e_i \otimes v_i \right) \right) &= \mathrm{ad}_{V_2} \left(e \otimes \left(\sum_{i=1}^{r} e_i \otimes f(v_i) \right) \right) \\
&= e \left(\sum_{i=1}^{r} e_i \otimes f(v_i) \right) \\
&= \sum_{i=1}^{r} ee_i \otimes f(v_i)
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{id} \otimes f) \circ \mathrm{ad}_{V_1} \left(e \otimes \left(\sum_{i=1}^{r} e_i \otimes v_i \right) \right) &= (\mathrm{id} \otimes f) \left(\sum_{i=1}^{r} ee_i \otimes v_i \right) \\
&= \sum_{i=1}^{r} ee_i \otimes f(v_i).
\end{aligned}$$

Restricting (4.4) to the fixed points of the operators φ_i, we obtain that the lower square of (4.3) commutes, as desired. $\square$

In the following two lemmas we explain why it is sufficient to show that $\mathbb{D}$ is essentially surjective. We also note that the space $E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}(D)$ comes equipped with Frobenius operators φ_i coming from the ones on $E_{\Delta_n}^{\mathrm{sep}}$, as well as a diagonal $G_{\Delta_n,L}$-action.

Lemma 4.4. *Assume that the functor*

$$\mathbb{D} : \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L}) \to \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$$

is essentially surjective. Then

(i) Every object of $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ is a finitely generated free module over E_{Δ_n}.

(ii) For $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ we have that $\mathbb{V}(D) \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$, hence $\mathbb{V}$ is a well defined functor

$$\mathbb{V} : \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n}) \to \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L}).$$

(iii) The natural map

$$\begin{aligned}
\mathrm{ad}_D : E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}(D) &\longrightarrow E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} D \\
e \otimes v &\longmapsto ev
\end{aligned}$$

is an isomorphism which commutes with the action of $G_{\Delta_n,L}$ and with the Frobenius operators φ_i on both sides.

(iv) The space $E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} D$ has an $E_{\Delta_n}^{\mathrm{sep}}$-basis fixed by the operators φ_i.

Proof. (i) This is immediate by the essential surjectivity of $\mathbb{D}$ and Corollary 4.2 (i).

(ii) By the essential surjectivity of $\mathbb{D}$, there exists a representation $V \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$ and an isomorphism

$$f : D \to \mathbb{D}(V)$$

in $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$. Therefore $\mathbb{V}(f) : \mathbb{V}(D) \to \mathbb{V}(\mathbb{D}(V))$ is an isomorphism. By Corollary 4.3 we know that $\mathbb{V}(\mathbb{D}(V)) \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$ and thus

$$\mathbb{V}(D) \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$$

by the $G_{\Delta_n,L}$-equivariance of $\mathbb{V}(f)$.

(iii) Consider the diagram

$$\begin{tikzcd}
E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}(D) \arrow[r, "\mathrm{ad}_D"] & E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} D \arrow[d, "\mathrm{id} \otimes f"] \\
E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}(\mathbb{D}(V)) \arrow[u, "\mathrm{id} \otimes \mathbb{V}(f^{-1})"] \arrow[r, "\mathrm{ad}_{\mathbb{D}(V)}"] & E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V) \arrow[d, "\mathrm{ad}_V"] \\
E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} V \arrow[u, "\mathrm{id} \otimes \mathrm{adj}_V"] \arrow[r, equal] & E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} V.
\end{tikzcd}$$

We claim that it commutes. To check that the upper square commutes, let

$$e \otimes \left(\sum_{i=1}^{r} e_i \otimes d_i \right) \in E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}(\mathbb{D}(V))$$

for $e, e_i \in E_{\Delta_n}^{\mathrm{sep}}$ and $d_i \in \mathbb{D}(V)$ when $1 \leq i \leq r$ such that

$$\sum_{i=1}^{r} e_i \otimes d_i \in \mathbb{V}(\mathbb{D}(V)) \subseteq E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} \mathbb{D}(V).$$

By direct computation

$$\begin{aligned}
(\mathrm{id} \otimes f) \circ \mathrm{ad}_D \circ (\mathrm{id} \otimes \mathbb{V}(f^{-1})) \left(e \otimes \left(\sum_{i=1}^{r} e_i \otimes d_i \right) \right) &= (\mathrm{id} \otimes f) \circ \mathrm{ad}_D \left(e \otimes \left(\sum_{i=1}^{r} e_i \otimes f^{-1}(d_i) \right) \right) \\
&= (\mathrm{id} \otimes f) \left(e \left(\sum_{i=1}^{r} e_i \otimes f^{-1}(d_i) \right) \right) \\
&= (\mathrm{id} \otimes f) \left(\sum_{i=1}^{r} e e_i \otimes f^{-1}(d_i) \right) \\
&= \sum_{i=1}^{r} e e_i \otimes d_i \\
&= \mathrm{ad}_{\mathbb{D}(V)} \left(e \otimes \left(\sum_{i=1}^{r} e_i \otimes d_i \right) \right).
\end{aligned}$$

To check that the lower square commutes, let $e \otimes w \in E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} V$. Then

$$\begin{aligned}
\mathrm{ad}_V \circ \mathrm{ad}_{\mathbb{D}(V)} \circ (\mathrm{id} \otimes \mathrm{adj}_V)(e \otimes w) &= \mathrm{ad}_V \circ \mathrm{ad}_{\mathbb{D}(V)} (e \otimes \mathrm{adj}_V(w)) \\
&= \mathrm{ad}_V(e\, \mathrm{adj}_V(w)) \\
&= \mathrm{ad}_V(e\, \mathrm{ad}_V^{-1}(\eta_V(w))) \\
&= e\eta_V(w) = e \otimes w.
\end{aligned}$$

Therefore

$$\mathrm{ad}_D = \left((\mathrm{id} \otimes \mathbb{V}(f^{-1})) \circ (\mathrm{id} \otimes \mathrm{adj}_V) \circ \mathrm{ad}_V \circ (\mathrm{id} \otimes f)\right)^{-1}$$

is a composition of bijective maps, making ad_D a bijection as well. The fact that ad_D respects the Galois action and the Frobenius operators is established in the same way as in the proof of Corollary 4.2 (iii). $\square$

Let $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$. Maintaining the assumption that the functor

$$\mathbb{D} : \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L}) \to \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$$

is essentially surjective, by Corollary 4.4 (i) we know that D is a free module over E_{Δ_n} with a trivial action from the group $H_{\Delta_n,L}$. Therefore, a basis of D over E_{Δ_n} extends to a basis of $E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} D$ over $E_{\Delta_n}^{\mathrm{sep}}$ fixed by the group $H_{\Delta_n,L}$. This implies that the map

$$\begin{aligned}
\mu_D : D &\to \left(E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} D\right)^{H_{\Delta_n,L}} \\
d &\mapsto 1 \otimes d
\end{aligned}$$

is an isomorphism by Corollary 3.41 (i). The map ad_D respects the Galois actions and the Frobenius operators on both sides, therefore it restricts to an isomorphism

$$\mathrm{ad}_D : \left(E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}(D)\right)^{H_{\Delta_n,L}} \to \left(E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} D\right)^{H_{\Delta_n,L}}.$$

We consider the isomorphism

$$\mathrm{adj}_D : D \xrightarrow{\mu_D} \left(E_{\Delta_n}^{\mathrm{sep}} \otimes_{E_{\Delta_n}} D\right)^{H_{\Delta_n,L}} \xrightarrow{\mathrm{ad}_D^{-1}} \left(E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}(D)\right)^{H_{\Delta_n,L}} = \mathbb{D}(\mathbb{V}(D)).$$

Lemma 4.5. *Assuming that the functor* $\mathbb{D} : \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L}) \to \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ *is essentially surjective, the maps* $\{\mathrm{adj}_D\}_{D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})}$ *give a natural isomorphism of functors*

$$\mathrm{id}_{\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})} \simeq \mathbb{D} \circ \mathbb{V}.$$

Proof. Let D_1 and D_2 be étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over E_{Δ_n} and f be a morphism between them. We need to prove that the following diagram

$$\begin{tikzcd}
D_1 \arrow[r, "f"] \arrow[d, "\mathrm{adj}_{D_1}"] & D_2 \arrow[d, "\mathrm{adj}_{D_2}"] \\
\mathbb{D}(\mathbb{V}(D_1)) \arrow[r, "\mathbb{D}(\mathbb{V}(f))"] & \mathbb{D}(\mathbb{V}(D_2))
\end{tikzcd}$$

commutes. We break our diagram into

$$\begin{array}{ccc} D_1 & \xrightarrow{\qquad\qquad f \qquad\qquad} & D_2 \\ \downarrow{\scriptstyle \mu_{D_1}} & & \downarrow{\scriptstyle \mu_{D_2}} \\ \left(E^{\mathrm{sep}}_{\Delta_n} \otimes_{E_{\Delta_n}} D_1\right)^{H_{\Delta_n,L}} & \xrightarrow{\mathrm{id}\otimes f} & \left(E^{\mathrm{sep}}_{\Delta_n} \otimes_{E_{\Delta_n}} D_2\right)^{H_{\Delta_n,L}} \\ \downarrow{\scriptstyle \mathrm{ad}^{-1}_{D_1}} & & \downarrow{\scriptstyle \mathrm{ad}^{-1}_{D_2}} \\ \mathbb{D}(\mathbb{V}(D_1)) & \xrightarrow{\mathbb{D}(\mathbb{V}(f))} & \mathbb{D}(\mathbb{V}(D_2)) \end{array} \tag{4.5}$$

and show that both squares commute. For the upper square, one computes that

$$(\mathrm{id}\otimes f)(\mu_{D_1}(d)) = (\mathrm{id}\otimes f)(1\otimes d) = 1 \otimes f(d) = \mu_{D_2}(f(d))$$

holds for $d \in D_1$. For the lower square, we show that the following diagram

$$\begin{array}{ccc} E^{\mathrm{sep}}_{\Delta_n} \otimes_{\kappa_L} \mathbb{V}(D_1) & \xrightarrow{\mathrm{id}\otimes \mathbb{V}(f)} & E^{\mathrm{sep}}_{\Delta_n} \otimes_{\kappa_L} \mathbb{V}(D_2) \\ \downarrow{\scriptstyle \mathrm{ad}_{D_1}} & & \downarrow{\scriptstyle \mathrm{ad}_{D_2}} \\ E^{\mathrm{sep}}_{\Delta_n} \otimes_{E_{\Delta_n}} D_1 & \xrightarrow{\mathrm{id}\otimes f} & E^{\mathrm{sep}}_{\Delta_n} \otimes_{E_{\Delta_n}} D_2 \end{array} \tag{4.6}$$

commutes. To check this, let

$$e \otimes \left(\sum_{i=1}^{r} e_i \otimes d_i\right) \in E^{\mathrm{sep}}_{\Delta_n} \otimes_{\kappa_L} \mathbb{V}(D_1)$$

for $e, e_i \in E^{\mathrm{sep}}_{\Delta_n}$ and $d_i \in D_1$ when $1 \leq i \leq r$ such that

$$\sum_{i=1}^{r} e_i \otimes d_i \in \mathbb{V}(D_1) \subseteq E^{\mathrm{sep}}_{\Delta_n} \otimes_{E_{\Delta_n}} D_1.$$

By direct computation

$$\begin{aligned} \mathrm{ad}_{D_2} \circ (\mathrm{id}\otimes \mathbb{V}(f)) \left(e \otimes \left(\sum_{i=1}^{r} e_i \otimes d_i\right)\right) &= \mathrm{ad}_{D_2} \left(e \otimes \left(\sum_{i=1}^{r} e_i \otimes f(d_i)\right)\right) \\ &= e \left(\sum_{i=1}^{r} e_i \otimes f(d_i)\right) \\ &= \sum_{i=1}^{r} ee_i \otimes f(d_i) \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id}\otimes f) \circ \mathrm{ad}_{D_1} \left(e \otimes \left(\sum_{i=1}^{r} e_i \otimes d_i\right)\right) &= (\mathrm{id}\otimes f) \left(\sum_{i=1}^{r} ee_i \otimes d_i\right) \\ &= \sum_{i=1}^{r} ee_i \otimes f(d_i). \end{aligned}$$

Restricting (4.6) to the $H_{\Delta_n,L}$-invariants, we obtain that the lower square of (4.5) commutes and we are done. $\square$

We prove the essential surjectivity of $\mathbb{D}$ by induction on n. For $n = 1$, this is Corollary 3.2.7 in [Sch17]. Suppose $n \geq 2$ and that the essential surjectivity of $\mathbb{D}$ holds true for all the positive integers up to $n - 1$.

4.2 The modules D^+ and $D^{+*}_{\overline{n}}$

We introduce the notion of boundedness for finitely generated modules over E_{Δ_n} equipped with the adic topology.

Definition 4.6. *Let D be a finitely generated E_{Δ_n}-module equipped with the adic topology. Let $\{d_1, \ldots, d_r\}$ be a generating set of D over E_{Δ_n}. A subset $B \subseteq D$ is* bounded *if*

$$B \subseteq X^m_{\Delta_n} \left(E^+_{\Delta_n} d_1 + \ldots + E^+_{\Delta_n} d_r \right)$$

for some $m \in \mathbb{Z}$.

Remark 4.7. In the above definition, the choice of the finite generating set does not matter. Indeed, if $x_1, \ldots, x_k$ is another generating set of D, then clearly

$$\{x_1, \ldots, x_k\} \subseteq X^\ell_{\Delta_n} \left(E^+_{\Delta_n} d_1 + \ldots + E^+_{\Delta_n} d_r \right)$$

for some $\ell \in \mathbb{Z}$ and viceversa. The same argument also shows that every finitely generated $E^+_{\Delta_n}$-submodule of D is bounded.

For $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ define

$$D^+ := \{x \in D : \{\varphi^k(x) : k \geq 0\} \subseteq D \text{ is bounded }\}.$$

Example 4.8. If $D = E_{\Delta_n}$ is equipped with the usual $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-structure, then $D^+ = E^+_{\Delta_n}$ and $D^{++} = X_{\Delta_n} E^+_{\Delta_n}$.

Proposition 4.9. *D^+ is an $E^+_{\Delta_n}$-submodule of D such that*

(i) $D = D^+[X^{-1}_{\Delta_n}]$,

(ii) $\tau(D^+) \subseteq D^+$ *for every* $\tau \in \mathcal{T}_{+,\Delta_n,L}$,

(iii) D^+ *is finitely generated over* $E^+_{\Delta_n}$.

Proof. Suppose that $\{d_1, \ldots, d_r\}$ is a generating set of D and let

$$M := E^+_{\Delta_n} d_1 + \ldots + E^+_{\Delta_n} d_r.$$

Let $x, y \in D^+$. Then the sets $\{\varphi^k(x) : k \geq 0\}$ and $\{\varphi^k(y) : k \geq 0\}$ are bounded, meaning that they are both contained in $X^{-m}_{\Delta_n} M$ for a large enough $m \in \mathbb{N}_{\geq 0}$. Since $X^{-m}_{\Delta_n} M$ is an $E^+_{\Delta_n}$-module, it follows that

$$\{\varphi^k(x + y) : k \geq 0\} = \{\varphi^k(x) + \varphi^k(y) : k \geq 0\} \subseteq X^{-m}_{\Delta_n} M$$

and

$$\{\varphi^k(ax) : k \geq 0\} = \{a^{q^k}\varphi^k(x) : k \geq 0\} \subseteq X_{\Delta_n}^{-m} M$$

for $a \in E_{\Delta_n}^+$. Therefore D^+ is an $E_{\Delta_n}^+$-submodule of D.

We also have that

$$\varphi^k(X_{\Delta_n} x) = X_{\Delta_n}^{q^k}\varphi^k(x) \in X_{\Delta_n}^{q^k - m} M$$

which shows that $\lim_{k\to\infty} \varphi^k(X_{\Delta_n} x) = 0$. It follows that $X_{\Delta_n} D^+ \subseteq D^{++}$ and Proposition 2.39 shows that D^{++} is a Noetherian $E_{\Delta_n}^+$-module, which implies that D^+ is finitely generated over $E_{\Delta_n}^+$, proving (iii). It is also clear that $D^{++} \subseteq D^+$ and Corollary 2.36 implies that $D^+[X_{\Delta_n}^{-1}] = D$ which proves (i).

To prove (ii), note that since $\{x, \varphi(x), \ldots\}$ is bounded, its subset $\{\varphi^k(x), \varphi^{k+1}(x), \ldots\}$ is bounded as well, which means that $\varphi^k(x) \in D^+$ for every $k \in \mathbb{N}_{\geq 0}$. Since D^+ is a finitely generated $E_{\Delta_n}^+$-submodule of D, so is $E_{\Delta_n}^+\tau(D^+)$. By Remark 4.7 it follows that $E_{\Delta_n}^+\tau(D^+)$ is a bounded subset of D. Hence

$$\{\tau(x), \varphi(\tau(x)), \varphi^2(\tau(x)), \ldots\} = \{\tau(x), \tau(\varphi(x)), \tau(\varphi^2(x)), \ldots\} \subseteq E_{\Delta_n}^+\tau(D^+)$$

is bounded and $\tau(x) \in D^+$, as desired. □

Lemma 4.10. *If D_1 and $D_2 \in \mathrm{Mod}^{ét}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$, then*

$$(D_1 \oplus D_2)^+ = D_1^+ \oplus D_2^+.$$

Proof. Suppose that $\{x_1, \ldots, x_k\}$ is a generating set of D_1 over E_{Δ_n} and $\{y_1, \ldots, y_\ell\}$ is a generating set of D_2 over E_{Δ_n}. Then $\{(x_1, 0), \ldots, (x_k, 0), (0, y_1), \ldots, (0, y_\ell)\}$ is a generating set of $D_1 \oplus D_2$ over E_{Δ_n}. Let $M_1 := \sum_{i=1}^{k} E_{\Delta_n}^+ x_i$, $M_2 := \sum_{j=1}^{\ell} E_{\Delta_n}^+ y_j$ and

$$M := \sum_{i=1}^{k} E_{\Delta_n}^+(x_i, 0) + \sum_{j=1}^{\ell} E_{\Delta_n}^+(0, y_j).$$

Let $(d_1, d_2) \in D_1^+ \oplus D_2^+$. Then

$$\{\varphi^k(d_1) : k \geq 0\} \subseteq X_{\Delta_n}^{-m} M_1$$

and

$$\{\varphi^k(d_2) : k \geq 0\} \subseteq X_{\Delta_n}^{-m} M_2$$

for a large enough $m \in \mathbb{N}_{\geq 0}$. Then

$$\begin{aligned}\{(d_1, d_2), \varphi(d_1, d_2), \varphi^2(d_1, d_2), \ldots\} &= \{(d_1, d_2), (\varphi(d_1), \varphi(d_2)), (\varphi^2(d_1), \varphi^2(d_2)), \ldots\} \\ &\subseteq X_{\Delta_n}^{-m} M\end{aligned}$$

and thus is bounded. This shows that $D_1^+ \oplus D_2^+ \subseteq (D_1 \oplus D_2)^+$. Now consider $(d_1, d_2) \in (D_1 \oplus D_2)^+$. Then

$$\{(d_1, d_2), \varphi(d_1, d_2), \varphi^2(d_1, d_2), \ldots\} = \{(d_1, d_2), (\varphi(d_1), \varphi(d_2)), (\varphi^2(d_1), \varphi^2(d_2)), \ldots\}$$

is bounded, thus it is contained in $X_{\Delta_n}^{-s}M$ for some $s \in \mathbb{N}_{\geq 0}$. Since in fact

$$M = \sum_{i=1}^{k} E_{\Delta_n}^{+}(x_i, 0) \bigoplus \sum_{j=1}^{\ell} E_{\Delta_n}^{+}(0, y_j)$$

is an internal direct sum, it follows that

$$\{d_1, \varphi(d_1), \ldots\} \subseteq X_{\Delta_n}^{-s} M_1$$

and

$$\{d_2, \varphi(d_2), \ldots\} \subseteq X_{\Delta_n}^{-s} M_2.$$

In other words, $d_1 \in D_1^{+}$ and $d_2 \in D_2^{+}$, as desired. □

We now introduce the module

$$D_{\overline{n}}^{+} \coloneqq D^{+}[X_{\Delta_{n-1}}^{-1}].$$

By Proposition 4.9 (iii) $D_{\overline{n}}^{+}$ is a finitely generated module over the Noetherian ring

$$E_{\overline{n}}^{+} \coloneqq E_{\Delta_n}^{+}[X_{\Delta_{n-1}}^{-1}].$$

Lemma 4.11. *For $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$, the module $D_{\overline{n}}^{+}/D^{+}$ is X_n-torsion free in the sense that if $X_n d \in D^{+}$ for some $d \in D_{\overline{n}}^{+}$, then $d \in D^{+}$.*

Proof. Since $d \in D_{\overline{n}}^{+}$, it follows that $X_{\Delta_{n-1}}^{m} d \in D^{+}$ for some $m \in \mathbb{N}_{\geq 0}$.

Case 1. Suppose that D is a free module over E_{Δ_n}. Let $d_1, \ldots, d_r$ be a basis of D over E_{Δ_n} and $M = E_{\Delta_n}^{+} d_1 + \ldots + E_{\Delta_n}^{+} d_r$. For $k \in \mathbb{N}_{\geq 0}$, write

$$\varphi^k(d) = \sum_{j=1}^{r} c_{jk} d_j$$

for some $c_{jk} \in E_{\Delta_n}$. Since $X_n d \in D^{+}$, it follows that the set

$$\{\varphi^k(X_n d)\}_{k \geq 0} = \{X_n^{q^k} \varphi^k(d)\}_{k \geq 0} = \{\sum_{j=1}^{r} X_n^{q^k} c_{jk} d_j\}_{k \geq 0}$$

is bounded. Therefore it is contained in $X_{\Delta_n}^{-m_1} M$ for some large enough $m_1 \in \mathbb{N}_{\geq 1}$ and it follows that the X_i-valuations of the elements $X_n^{q^k} c_{jk}$ are at least $-m_1$ for every $i \in \Delta_n$. Therefore the X_i-valuations of the elements c_{jk} are at least $-m_1$ for every $i \in \Delta_{n-1}$. Similarly, since $X_{\Delta_{n-1}}^{m} d \in D^{+}$ it follows that the set

$$\{\varphi^k(X_{\Delta_{n-1}}^{m} d)\}_{k \geq 0} = \{X_{\Delta_{n-1}}^{mq^k} \varphi^k(d)\}_{k \geq 0} = \{\sum_{j=1}^{r} X_{\Delta_{n-1}}^{mq^k} c_{jk} d_j\}_{k \geq 0}$$

is bounded. Therefore it is contained in $X_{\Delta_n}^{-m_2}M$ for some large enough $m_2 \in \mathbb{N}_{\geq 1}$ and it follows that the X_n-valuations of the elements $X_{\Delta_{n-1}}^{mq^k} c_{jk}$ are at least $-m_2$. Therefore the X_n-valuations of the elements c_{jk} are also at least $-m_2$.

Therefore $c_{jk} \in X_{\Delta_n}^{-\max\{m_1,m_2\}} E_{\Delta_n}^+$, which implies that $\{d, \varphi(d), \ldots\}$ is bounded, as desired.

Case 2. In the general case, using that D is stably-free, let $k_1 \in \mathbb{N}_{\geq 0}$ such that

$$D_1 = D \oplus E_{\Delta_n}^{\oplus k_1} \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$$

is a finite free module over E_{Δ_n}. By Example 4.8 and Lemma 4.10 we have that $D_1^+ = D^+ \oplus (E_{\Delta_n}^+)^{\oplus k_1}$. Inverting $X_{\Delta_{n-1}}$ it also follows that

$$(D_1)_{\overline{n}}^+ = D_{\overline{n}}^+ \oplus (E_{\overline{n}}^+)^{\oplus k_1}.$$

Then by hypothesis $(d, 0) \in (D_1)_{\overline{n}}^+$ and $X_n(d, 0) \in D_1^+$. By Case 1 it follows that

$$(d, 0) \in D_1^+ = D^+ \oplus (E_{\Delta_n}^+)^{\oplus k_1}$$

meaning that $d \in D^+$, as desired. $\square$

Let $\mathcal{T}_{+,\Delta_{n-1},L} \subseteq \mathcal{T}_{+,\Delta_n,L}$ be the submonoid generated by φ_j for $j \in \Delta_{n-1}$ and $\Gamma_{\Delta_{n-1},L}$.

Lemma 4.12. *Suppose that $D \in \mathrm{Mod}^{ét}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ is a free module of rank one generated by z. Then for any $\tau \in \mathcal{T}_{+,\Delta_{n-1},L}$ we have that*

$$\tau(z) = a_\tau z,$$

where $a_\tau \in (E_{\overline{n}}^+)^\times$.

Proof. Suppose that $\varphi_n(z) = a_n z$ and $\tau(z) = a_\tau z$. Since D is étale, it follows that both a_n and a_τ are in $E_{\Delta_n}^\times$. To show that $a_\tau \in (E_{\overline{n}}^+)^\times$ we are left to prove that $\mathrm{val}_{X_n}(a_\tau) = 0$ because $\mathrm{val}_{X_n}(a_\tau) \geq 0$ implies that $a_\tau \in E_{\overline{n}}^+$, while $\mathrm{val}_{X_n}(a_\tau^{-1}) \geq 0$ shows that $a_\tau^{-1} \in E_{\overline{n}}^+$.

We note that

$$\begin{aligned}
\tau(\varphi_n(z)) &= \tau(a_n z) = \tau(a_n)\tau(z) = \tau(a_n)a_\tau z, \\
\varphi_n(\tau(z)) &= \varphi_n(a_\tau z) = \varphi_n(a_\tau)\varphi_n(z) = \varphi_n(a_\tau)a_n z.
\end{aligned}$$

Using that τ and φ_n commute and that z is a basis element, we conclude that $\tau(a_n)a_\tau = \varphi_n(a_\tau)a_n$. Therefore

$$\begin{aligned}
\mathrm{val}_{X_n}(a_n) + \mathrm{val}_{X_n}(a_\tau) &= \mathrm{val}_{X_n}(\tau(a_n)) + \mathrm{val}_{X_n}(a_\tau) \\
&= \mathrm{val}_{X_n}(\tau(a_n)a_\tau) \\
&= \mathrm{val}_{X_n}(\varphi_n(a_\tau)a_n) \\
&= q\mathrm{val}_{X_n}(a_\tau) + \mathrm{val}_{X_n}(a_n)
\end{aligned}$$

where the first equality holds because $\tau \in \mathcal{T}_{+,\Delta_{n-1},L}$. This implies that $\mathrm{val}_{X_n}(a_\tau) = 0$ and the conclusion follows. $\square$

Lemma 4.13. *There exists an integer $k = k(D) > 0$ such that for any $\tau \in \mathcal{T}_{+,\Delta_{n-1},L}$ we have*
$$X_n^k D_{\overline{n}}^+ \subseteq E_{\overline{n}}^+ \tau(D_{\overline{n}}^+).$$

Proof. Assume first that D is a free module over E_{Δ_n} with basis $e_1, \ldots, e_r$. By Proposition 4.9 we can further assume that $e_1, \ldots, e_r$ are contained in D^+. Let
$$M \coloneqq E_{\Delta_n}^+ e_1 + \ldots + E_{\Delta_n}^+ e_r$$
and $M_{\overline{n}} \coloneqq M[X_{\Delta_{n-1}}^{-1}]$. By assumption $M \subseteq D^+$ and $M_{\overline{n}} \subseteq D_{\overline{n}}^+$. Since D^+ is finitely generated over $E_{\Delta_n}^+$ by Proposition 4.9 (iii), it follows that $D^+ \subseteq X_{\Delta_n}^{-k_0} M$ for some $k_0 \in \mathbb{N}_{>0}$. Inverting $X_{\Delta_{n-1}}$, it follows that
$$D_{\overline{n}}^+ \subseteq X_n^{-k_0} M_{\overline{n}}.$$
Let $\tau \in \mathcal{T}_{+,\Delta_{n-1},L}$ arbitrary and let A_τ denote the matrix of τ with respect to the basis $e_1, \ldots, e_r$. Since $e_1, \ldots, e_r \in D^+$, by Proposition 4.9 (ii) it follows that $\tau(e_1), \ldots, \tau(e_r) \in D^+$. Therefore $\tau(e_1), \ldots, \tau(e_r) \in X_{\Delta_n}^{-k_0} M$, meaning that the entries of A_τ lie in $X_{\Delta_n}^{-k_0} E_{\Delta_n}^+$. By Example 2.28, we know that $\det(D) \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ and is free over E_{Δ_n} of rank one. Therefore one can apply Lemma 4.12 for the basis element $e_1 \wedge \ldots \wedge e_r \in \det(D)$ and it follows that
$$\mathrm{val}_{X_n}(\det A_\tau) = 0.$$
Using the formula for A_τ^{-1} involving the $(r-1) \times (r-1)$ minors, it follows that all the entries of A_τ^{-1} lie in $X_n^{-(r-1)k_0} E_{\overline{n}}^+$. Using that $(e_1 \ldots e_r) = (\tau(e_1) \ldots \tau(e_r)) A_\tau^{-1}$, we obtain that
$$X_n^{k_0} D_{\overline{n}}^+ \subseteq M_{\overline{n}} \subseteq X_n^{-(r-1)k_0} E_{\overline{n}}^+ \tau(M_{\overline{n}}) \subseteq X_n^{-(r-1)k_0} E_{\overline{n}}^+ \tau(D_{\overline{n}}^+).$$
Therefore $X_n^{rk_0} D_{\overline{n}}^+ \subseteq E_{\overline{n}}^+ \tau(D_{\overline{n}}^+)$ and we can take $k = rk_0$ which depends only on D and not on the chosen $\tau \in \mathcal{T}_{+,\Delta_{n-1},L}$.

In the general case $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ is stably-free, so let $\ell \in \mathbb{N}$ such that
$$D_1 = D \oplus E_{\Delta_n}^{\oplus \ell} \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$$
is a free module over E_{Δ_n}. From the first case, there exists an integer k independent of τ such that
$$X_n^k (D_1)_{\overline{n}}^+ = X_n^k (D_{\overline{n}}^+ \oplus (E_{\overline{n}}^+)^{\oplus \ell_0}) \subseteq E_{\overline{n}}^+ \tau(D_{\overline{n}}^+ \oplus (E_{\overline{n}}^+)^{\oplus \ell_0}) = E_{\overline{n}}^+ (\tau(D_{\overline{n}}^+) \oplus \tau((E_{\overline{n}}^+)^{\oplus \ell_0}))$$
which implies that $X_n^k D_{\overline{n}}^+ \subseteq E_{\overline{n}}^+ \tau(D_{\overline{n}}^+)$, as desired. □

We let
$$D_{\overline{n}}^{+*} \coloneqq \bigcap_{\tau \in \mathcal{T}_{+,\Delta_{n-1},L}} E_{\overline{n}}^+ \tau(D_{\overline{n}}^+).$$
As $D_{\overline{n}}^{+*} \subseteq D_{\overline{n}}^+$ and $E_{\overline{n}}^+$ is a Noetherian ring, it follows that $D_{\overline{n}}^{+*}$ is a finitely generated module over $E_{\overline{n}}^+$. On the other hand, by Lemma 4.13 it follows that $X_n^{k(D)} D_{\overline{n}}^+ \subseteq D_{\overline{n}}^{+*}$, therefore $D_{\overline{n}}^{+*}[X_n^{-1}] \simeq D$.

Proposition 4.14. *For $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ we have that*

(i) $\tau(D^{+*}_{\overline{n}}) \subseteq D^{+*}_{\overline{n}}$ for all $\tau \in \mathcal{T}_{+,\Delta_n,L}$,

(ii) $D^{+*}_{\overline{n}}$ is an étale $\mathcal{T}_{+,\Delta_{n-1},L}$-module over $E^+_{\overline{n}}$ in the sense that the map

$$\begin{aligned} \tau^{\mathrm{lin}} : E^+_{\overline{n}} \otimes_{\tau, E^+_{\overline{n}}} D^{+*}_{\overline{n}} &\longrightarrow D^{+*}_{\overline{n}} \\ e \otimes d &\mapsto e\tau(d) \end{aligned}$$

is a bijection for all $\tau \in \mathcal{T}_{+,\Delta_{n-1},L}$.

Proof. (i) Inverting $X_{\Delta_{n-1}}$ in Proposition 4.9 (ii) we know that $\tau(D^+_{\overline{n}}) \subseteq D^+_{\overline{n}}$. Let $\tau_0 \in \mathcal{T}_{+,\Delta_{n-1},L}$. Then, because $\mathcal{T}_{+,\Delta_n,L}$ is a commutative monoid

$$\begin{aligned} \tau(E^+_{\overline{n}}\tau_0(D^+_{\overline{n}})) &= \tau(E^+_{\overline{n}})\tau(\tau_0(D^+_{\overline{n}})) \\ &\subseteq E^+_{\overline{n}}\tau_0(\tau(D^+_{\overline{n}})) \\ &\subseteq E^+_{\overline{n}}\tau_0(D^+_{\overline{n}}). \end{aligned}$$

Thus $\tau(D^{+*}_{\overline{n}}) = \tau\left(\bigcap_{\tau_0 \in \mathcal{T}_{+,\Delta_{n-1},L}} E^+_{\overline{n}}\tau_0(D^+_{\overline{n}})\right) \subseteq \bigcap_{\tau_0 \in \mathcal{T}_{+,\Delta_{n-1},L}} E^+_{\overline{n}}\tau_0(D^+_{\overline{n}}) = D^{+*}_{\overline{n}}$, as desired.

(ii) Suppose that $\tau \in \mathcal{T}_{+,\Delta_{n-1},L}$ can be written as

$$\tau = \prod_{i \in \Delta_{n-1}} \varphi_i^{s_i}\sigma_i$$

for $\sigma_i \in \Gamma_{i,L}$ and $s_i \in \mathbb{N}_{\geq 0}$. Then $E^+_{\Delta_n}$ is a free module over the subring $\tau(E^+_{\Delta_n})$ with basis $\mathfrak{B} = \{\prod_{i \in \Delta_{n-1}} X_i^{j_i}\}_{0 \leq j_i \leq q^{s_i}-1}$. Then $\mathfrak{B}$ is also a basis of E_{Δ_n} over $\tau(E_{\Delta_n})$ and of $E^+_{\overline{n}}$ over $\tau(E^+_{\overline{n}})$.

Let $\tau_0 \in \mathcal{T}_{+,\Delta_{n-1},L}$. Since $E^+_{\overline{n}}$ is free over $\tau(E^+_{\overline{n}})$, it follows $E^+_{\overline{n}}$ is a flat $\tau(E^+_{\overline{n}})$-module, therefore the natural map

$$\begin{aligned} E^+_{\overline{n}} \otimes_{\tau, E^+_{\overline{n}}} E^+_{\overline{n}}\tau_0(D^+_{\overline{n}}) &\longrightarrow E^+_{\overline{n}} \otimes_{\tau, E^+_{\overline{n}}} D \\ e \otimes d &\longmapsto e \otimes d \end{aligned}$$

induced by the inclusion $E^+_{\overline{n}}\tau_0(D^+_{\overline{n}}) \subseteq D$ is injective. Since $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ we know that

$$\tau^{\mathrm{lin}} : E_{\Delta_n} \otimes_{\tau, E_{\Delta_n}} D \longrightarrow D$$

is bijective. Since $\mathfrak{B}$ is a common basis of E_{Δ_n} over $\tau(E_{\Delta_n})$ and of $E^+_{\overline{n}}$ over $\tau(E^+_{\overline{n}})$ it follows that the map

$$\begin{aligned} \tau^{\mathrm{lin}} : E^+_{\overline{n}} \otimes_{\tau, E^+_{\overline{n}}} D &\longrightarrow D \\ e \otimes d &\longmapsto e\tau(d) \end{aligned}$$

is bijective as well. Indeed, we have a commutative diagram

$$\begin{tikzcd}
E_{\overline{n}}^{+} \otimes_{\tau, E_{\overline{n}}^{+}} D \arrow[r, "\tau^{\mathrm{lin}}"] \arrow[d, "\psi"] & D \arrow[dd, equal] \\
D^{\oplus \mathfrak{B}} \arrow[d, "\psi'"] & \\
E_{\Delta_n} \otimes_{\tau, E_{\Delta_n}} D \arrow[r, "\tau^{\mathrm{lin}}"] & D
\end{tikzcd} \tag{4.7}$$

where the map $\psi : E_{\overline{n}}^{+} \otimes_{\tau, E_{\overline{n}}^{+}} D \to D^{\oplus \mathfrak{B}}$ is defined by

$$\psi\left(\left(\sum_{\beta \in \mathfrak{B}} \tau(c_\beta)\beta\right) \otimes d\right) = (c_\beta d)_{\beta \in \mathfrak{B}},$$

while the map $\psi' : D^{\oplus \mathfrak{B}} \to E_{\Delta_n} \otimes_{\tau, E_{\Delta_n}} D$ is defined by

$$\psi'(d_\beta)_{\beta \in \mathfrak{B}} = \sum_{\beta \in \mathfrak{B}} \beta \otimes d_\beta.$$

Then ψ and ψ' are additive bijections making (4.7) commute. Its lower horizontal and vertical arrows are bijections, which forces the top horizontal arrow to be a bijection as well. Using the flatness of $E_{\overline{n}}^{+}$ over $\tau(E_{\overline{n}}^{+})$ it follows that the map

$$\begin{aligned}
E_{\overline{n}}^{+} \otimes_{\tau, E_{\overline{n}}^{+}} E_{\overline{n}}^{+} \tau_0(D_{\overline{n}}^{+}) &\longrightarrow D \\
e \otimes d &\longmapsto e\tau(d)
\end{aligned}$$

is injective with image $E_{\overline{n}}^{+}\tau(\tau_0(D_{\overline{n}}^{+}))$. Since $E_{\overline{n}}^{+}$ is finite free over $\tau(E_{\overline{n}}^{+})$ it follows that

$$\begin{aligned}
E_{\overline{n}}^{+} \otimes_{\tau, E_{\overline{n}}^{+}} D_{\overline{n}}^{+*} &= E_{\overline{n}}^{+} \otimes_{\tau, E_{\overline{n}}^{+}} \left(\bigcap_{\tau_0 \in \mathcal{T}_{+, \Delta_{n-1}, L}} E_{\overline{n}}^{+} \tau_0(D_{\overline{n}}^{+})\right) \\
&= \bigcap_{\tau_0 \in \mathcal{T}_{+, \Delta_{n-1}, L}} \left(E_{\overline{n}}^{+} \otimes_{\tau, E_{\overline{n}}^{+}} E_{\overline{n}}^{+} \tau_0(D_{\overline{n}}^{+})\right)
\end{aligned}$$

which means that

$$\begin{aligned}
E_{\overline{n}}^{+} \otimes_{\tau, E_{\overline{n}}^{+}} D_{\overline{n}}^{+*} &\longrightarrow D \\
e \otimes d &\longmapsto e\tau(d)
\end{aligned}$$

is injective with image $\bigcap_{\tau_0 \in \mathcal{T}_{+, \Delta_{n-1}, L}} E_{\overline{n}}^{+}\tau(\tau_0(D_{\overline{n}}^{+})) = D_{\overline{n}}^{+*}$, as desired. □

4.3 The module D_n

Consider the κ_L-algebra $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!)$. The groups $G_{\Delta_{n-1},L}$ and $\Gamma_{n,L}$ act on $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!)$ using the existing actions of $G_{\Delta_{n-1},L}$ on $E^{\mathrm{sep}}_{\Delta_{n-1}}$ and of $\Gamma_{n,L}$ on E_n. Similarly, for $i \in \Delta_{n-1}$, we have a Frobenius operator φ_i on $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!)$ using the existing operators on $E^{\mathrm{sep}}_{\Delta_{n-1}}$, as well as an operator φ_n using the existing one on E_n. Note that these group actions and operators preserve $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]$.

For $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$, the groups $G_{\Delta_{n-1},L}$ and $\Gamma_{n,L}$ act diagonally on the $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!)$-module $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D$, where $G_{\Delta_{n-1},L}$ acts on D through its quotient $\Gamma_{\Delta_{n-1},L}$. Also, for every $i \in \Delta_n$ there exists an operator φ_i that acts diagonally on $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D$ using the existing φ_i operators on $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!)$ and D. Inside $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D$ consider the subspace

$$D_n := \bigcap_{j \in \Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D\right)^{\varphi_j = \mathrm{id}}.$$

By Corollary 3.45 we know that D_n is a $\kappa_L(\!(X_n)\!)$-vector space. In addition, D_n is equipped with a φ_n operator, as well as an action from $G_{\Delta_{n-1},L}$ and $\Gamma_{n,L}$. Later we will also show that D_n is a finitely generated étale $(\varphi_n, \Gamma_{n,L})$-module over $\kappa_L(\!(X_n)\!)$ and that the map

$$\begin{aligned} E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{\kappa_L(\!(X_n)\!)} D_n &\longrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D \\ e \otimes d &\longmapsto ed \end{aligned} \tag{4.8}$$

is an isomorphism. In addition to $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D$, we also consider the module $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$. By Proposition 4.14 (i), the latter also comes equipped with diagonal actions of $G_{\Delta_{n-1},L}$ and $\Gamma_{n,L}$, as well as φ_i operators for $i \in \Delta_n$. Then

$$\mathbb{W}(D_n) := \bigcap_{i \in \Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i = \mathrm{id}}$$

is a $\kappa_L[\![X_n]\!]$-submodule of $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$ that is preserved by the actions of $G_{\Delta_{n-1},L}$ and $\Gamma_{n,L}$ as well as by the operator φ_n.

Lemma 4.15. *For any integer $r > 0$,*

(i) The operators φ_j act naturally on $E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r)$ and

$$\bigcap_{j \in \Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r)\right)^{\varphi_j = \mathrm{id}} = \kappa_L[X_n]/(X_n^r).$$

(ii) We have a natural isomorphism of functors

$$E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{\kappa_L[X_n]/X_n^r \kappa_L[X_n]} - \simeq E^{\mathrm{sep}}_{\Delta_{n-1}} \otimes_{\kappa_L} -$$

from the category of $\kappa_L[X_n]/X_n^r\kappa_L[X_n]$-modules to the category of $E^{\mathrm{sep}}_{\Delta_{n-1}}$-modules.

Proof. (i) Since the action of each φ_j on $E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!]$ is coefficientwise with respect to the powers of X_n, this is immediate from Corollary 3.45.

(ii) Using that $E^{\text{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \simeq E^{\text{sep}}_{\Delta_{n-1}} \otimes_{\kappa_L} \kappa_L[X_n]/(X_n^r)$, the conclusion follows. □

Lemma 4.16. *(i) We have an isomorphism*

$$E^+_{\overline{n}}/X_n^r E^+_{\overline{n}} \simeq E_{\Delta_{n-1}}[X_n]/X_n^r E_{\Delta_{n-1}}[X_n].$$

(ii) We have a natural isomorphism of functors

$$E^{\text{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}/X_n^r E^+_{\overline{n}}} - \simeq E^{\text{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} -$$

from the category of $E^+_{\overline{n}}/X_n^r E^+_{\overline{n}}$-modules to the category of $E^{\text{sep}}_{\Delta_{n-1}}$-modules.

(iii) We have a natural isomorphism of functors

$$E^{\text{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}} - \simeq E^{\text{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} -\right)$$

from the category of $E^+_{\overline{n}}$-modules to the category of $E^{\text{sep}}_{\Delta_{n-1}}$-modules.

(iv) We have a natural isomorphism of functors

$$E^{\text{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} - \simeq E^{\text{sep}\,+}_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} -\right)$$

from the category of $E^+_{\Delta_n}$-modules to the category of $E^{\text{sep}\,+}_{\Delta_{n-1}}$-modules.

Proof. (i) It is easy to observe that both sides are free modules over $E_{\Delta_{n-1}}$ with the basis elements $1, X_n, \ldots, X_n^{r-1}$.

(ii) Since $E^{\text{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \simeq E^{\text{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} E_{\Delta_{n-1}}[X_n]/(X_n^r)$, by part (i) the conclusion follows.

(iii) Let M be an $E^+_{\overline{n}}$-module. Then we have that

$$\begin{aligned}
E^{\text{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}} M &\simeq \left(E^{\text{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} E_{\Delta_{n-1}}[X_n]/(X_n^r)\right) \otimes_{E^+_{\overline{n}}} M \\
&\simeq E^{\text{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} \left(E^+_{\overline{n}}/X_n^r E^+_{\overline{n}} \otimes_{E^+_{\overline{n}}} M\right) \\
&\simeq E^{\text{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} M/X_n^r M \\
&\simeq E^{\text{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} M\right),
\end{aligned}$$

where for the second isomorphism we used the result of part (i).

(iv) Let M be an $E^+_{\Delta_n}$-module. Then we have that

$$\begin{aligned}
E^{\text{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M &\simeq \left(E^{\text{sep}\,+}_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} E^+_{\Delta_{n-1}}[X_n]/(X_n^r)\right) \otimes_{E^+_{\Delta_n}} M \\
&\simeq E^{\text{sep}\,+}_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(E^+_{\Delta_n}/X_n^r E^+_{\Delta_n} \otimes_{E^+_{\Delta_n}} M\right) \\
&\simeq E^{\text{sep}\,+}_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} M/X_n^r M \\
&\simeq E^{\text{sep}\,+}_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} M\right),
\end{aligned}$$

where for the second isomorphism we used that

$$E^+_{\Delta_{n-1}}[X_n]/(X_n^r) \simeq E^+_{\Delta_{n-1}}[\![X_n]\!]/(X_n^r)$$

and $E^+_{\Delta_n} = E^+_{\Delta_{n-1}}[\![X_n]\!]$. $\square$

Lemma 4.17. *Let $(E'_j/E_j)_{j\in\Delta_{n-1}}$ be a collection of finite separable extensions.*

(i) We have a natural isomorphism of functors

$$E'^+_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} - \simeq E'^+_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} -\right)$$

from the category of $E^+_{\Delta_n}$-modules to the category of $E'^+_{\Delta_{n-1}}$-modules.

(ii) We have a natural isomorphism of functors

$$E'_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}/X_n^r E^+_{\overline{n}}} - \simeq E'_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} -$$

from the category of $E^+_{\overline{n}}/X_n^r E^+_{\overline{n}}$-modules to the category of $E'_{\Delta_{n-1}}$-modules.

Proof. (i) Let M be an $E^+_{\Delta_n}$-module. Then we have that

$$\begin{aligned} E'^+_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M &\simeq \left(E'^+_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} E^+_{\Delta_{n-1}}[X_n]/(X_n^r)\right) \otimes_{E^+_{\Delta_n}} M \\ &\simeq E'^+_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(E^+_{\Delta_n}/X_n^r E^+_{\Delta_n} \otimes_{E^+_{\Delta_n}} M\right) \\ &\simeq E'^+_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} M/X_n^r M \\ &\simeq E'^+_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} M\right), \end{aligned}$$

where for the second isomorphism we used that

$$E^+_{\Delta_{n-1}}[X_n]/(X_n^r) \simeq E^+_{\Delta_n}/X_n^r E^+_{\Delta_n}.$$

(ii) Let M be an $E^+_{\overline{n}}/X_n^r E^+_{\overline{n}}$-module. Then we have that

$$\begin{aligned} E'_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}/X_n^r E^+_{\overline{n}}} M &\simeq \left(E'_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} E_{\Delta_{n-1}}[X_n]/(X_n^r)\right) \otimes_{E^+_{\overline{n}}/X_n^r E^+_{\overline{n}}} M \\ &\simeq E'_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} \left(E^+_{\overline{n}}/X_n^r E^+_{\overline{n}} \otimes_{E^+_{\overline{n}}/X_n^r E^+_{\overline{n}}} M\right) \\ &\simeq E'_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} M, \end{aligned}$$

where for the second isomorphism we used Lemma 4.16 (i). $\square$

Lemma 4.18. *Let M be an $E^+_{\overline{n}}$-module that is X_n-torsion free, then*

$$\mathrm{Tor}_i^{E^+_{\overline{n}}}\left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r), M\right) = 0$$

for all integers $i, r > 0$.

Proof. Since M has no X_n-torsion, it is torsion free hence flat over $\kappa_L[\![X_n]\!]$ and thus

$$\mathrm{Tor}_i^{\kappa_L[\![X_n]\!]}(\kappa_L[X_n]/(X_n^r), M) = 0 \tag{4.9}$$

for all $i, r > 0$. Let $P_\bullet \longrightarrow M \longrightarrow 0$ be a projective resolution of M in $E_{\overline{n}}^+ - \mathrm{Mod}$. Applying $E_{\Delta_{n-1}}^{\mathrm{sep}}[X_n]/(X_n^r) \otimes_{E_{\overline{n}}^+} -$ to this resolution, the i-th homology group of

$$E_{\Delta_{n-1}}^{\mathrm{sep}}[X_n]/(X_n^r) \otimes_{E_{\overline{n}}^+} P_\bullet \longrightarrow 0$$

computes $\mathrm{Tor}_i^{E_{\overline{n}}^+}\left(E_{\Delta_{n-1}}^{\mathrm{sep}}[X_n]/(X_n^r), M\right)$ for all $i > 0$. By Lemma 4.16 (iii), it suffices to show that the complex

$$\begin{aligned} &E_{\Delta_{n-1}}^{\mathrm{sep}} \otimes_{E_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} P_\bullet\right) \longrightarrow \\ &\longrightarrow E_{\Delta_{n-1}}^{\mathrm{sep}} \otimes_{E_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} M\right) \longrightarrow 0 \end{aligned} \tag{4.10}$$

is exact. Since P_i is projective over $E_{\overline{n}}^+$, it is torsion free over $E_{\overline{n}}^+$ and thus torsion free over $\kappa_L[\![X_n]\!]$. Hence

$$P_\bullet \longrightarrow M \longrightarrow 0$$

is a resolution of flat modules in $\kappa_L[\![X_n]\!] - \mathrm{Mod}$. Since resolutions of flat modules also allow us to compute the Tor functors, by (4.9) it follows that

$$\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} P_\bullet \longrightarrow \kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} M \longrightarrow 0$$

is an exact complex. By Lemma 3.12 (i) applied for the set Δ_{n-1} we know that $E_{\Delta_{n-1}}^{\mathrm{sep}}$ is a flat $E_{\Delta_{n-1}}$-module, so it follows that (4.10) is also exact and we are done. $\square$

As an immediate corollary of Lemma 4.18 we can establish the relationship between D_n and $\mathbb{W}(D_n)$.

Lemma 4.19. *(i) The module $E_{\Delta_{n-1}}^{\mathrm{sep}}[\![X_n]\!] \otimes_{E_{\overline{n}}^+} D_{\overline{n}}^{+*}$ is X_n-torsion free.*

(ii) The $\kappa_L[\![X_n]\!]$-module $\mathbb{W}(D_n)$ is torsion free.

(iii) We have that

$$E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D \simeq \left(E_{\Delta_{n-1}}^{\mathrm{sep}}[\![X_n]\!] \otimes_{E_{\overline{n}}^+} D_{\overline{n}}^{+*}\right)[X_n^{-1}]$$

and the natural map

$$E_{\Delta_{n-1}}^{\mathrm{sep}}[\![X_n]\!] \otimes_{E_{\overline{n}}^+} D_{\overline{n}}^{+*} \to E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D$$

is an embedding.

(iv) We have that $D_n \simeq \mathbb{W}(D_n)[X_n^{-1}]$ and the natural map

$$\mathbb{W}(D_n) \to D_n$$

is an embedding.

Proof. (i) Since $E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!]$ has no X_n-torsion, we have an exact sequence

$$0 \longrightarrow E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!] \xrightarrow{X_n \cdot} E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!] \to E^{\text{sep}}_{\Delta_{n-1}}[X_n]/(X_n) \longrightarrow 0$$

which after tensoring by $D^{+*}_{\overline{n}}$ over $E^{+}_{\overline{n}}$ gives the exact sequence

$$\begin{aligned} &\mathrm{Tor}_1^{E^+_{\overline{n}}}\left(E^{\text{sep}}_{\Delta_{n-1}}[X_n]/(X_n), D^{+*}_{\overline{n}}\right) \longrightarrow E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \xrightarrow{X_n \cdot} \\ &E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \longrightarrow E^{\text{sep}}_{\Delta_{n-1}}[X_n]/(X_n) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \longrightarrow 0. \end{aligned}$$

Since D is an E_{Δ_n}-module, it has no X_n-torsion. Therefore $D^{+*}_{\overline{n}}$ being a submodule of D, has no X_n-torsion either. By Lemma 4.18 applied for $M = D^{+*}_{\overline{n}}$ we know that

$$\mathrm{Tor}_1^{E^+_{\overline{n}}}\left(E^{\text{sep}}_{\Delta_{n-1}}[X_n]/(X_n), D^{+*}_{\overline{n}}\right) = 0$$

and the conclusion follows.

(ii) This follows immediately from part (i).

(iii) We have the isomorphisms

$$\begin{aligned} \left(E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)[X_n^{-1}] &\simeq E^{\text{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \\ &\simeq E^{\text{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E^+_{\overline{n}}[X_n^{-1}]} D^{+*}_{\overline{n}}[X_n^{-1}] \\ &\simeq E^{\text{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D. \end{aligned}$$

By part (i) $E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$ has no X_n-torsion, therefore the natural map

$$E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \to \left(E^{\text{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)[X_n^{-1}]$$

is an embedding.

(iv) Since X_n is fixed by φ_j for every $j \in \Delta_{n-1}$, this follows by (iii). $\square$

Recall that $D^{+*}_{\overline{n}}$ is a finitely generated étale $\mathcal{T}_{+,\Delta_{n-1},L}$-module over $E^+_{\overline{n}}$ by Proposition 4.14 (ii). However, $D^{+*}_{\overline{n}}$ need not be an object of $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_{n-1}}, \Gamma_{\Delta_{n-1},L}, E_{\Delta_{n-1}})$ since it need not be finitely generated over $E_{\Delta_{n-1}}$. Instead, for all $r \in \mathbb{N}_{>0}$ consider

$$D^{+*}_{\overline{n},r} := D^{+*}_{\overline{n}}/X_n^r D^{+*}_{\overline{n}}.$$

Then $D^{+*}_{\overline{n},r}$ is a finitely generated module over

$$E^{+}_{\overline{n},r} := E^{+}_{\overline{n}}/X_n^r E^{+}_{\overline{n}}.$$

By Lemma 4.16 (i), $D^{+*}_{\overline{n},r}$ is then finitely generated over $E_{\Delta_{n-1}}[X_n]/(X_n^r)$ and thus finitely generated over $E_{\Delta_{n-1}}$. We can now show:

Proposition 4.20. *$D^{+*}_{\overline{n},r}$ is an étale $(\varphi_{\Delta_{n-1}}, \Gamma_{\Delta_{n-1},L})$-module over $E_{\Delta_{n-1}}$ for every positive integer r.*

Proof. We need to show that for each $i \in \Delta_{n-1}$ the map

$$\begin{aligned} \varphi_i^{\mathrm{lin}} : E_{\Delta_{n-1}} \otimes_{\varphi_i, E_{\Delta_{n-1}}} D^{+*}_{\overline{n},r} &\longrightarrow D^{+*}_{\overline{n},r} \\ e \otimes d &\longmapsto e\varphi_i(d) \end{aligned}$$

is bijective, where $d \in D^{+*}_{\overline{n},r}$ and $e \in E_{\Delta_{n-1}}$. We factor φ_i^{lin} through the commutative triangle

$$\begin{tikzcd} E_{\Delta_{n-1}} \otimes_{\varphi_i, E_{\Delta_{n-1}}} D^{+*}_{\overline{n},r} \arrow[rr, "\varphi_i^{\mathrm{lin}}"] \arrow[dr, "\psi"'] & & D^{+*}_{\overline{n},r} \\ & E^{+}_{\overline{n}} \otimes_{\varphi_i, E^{+}_{\overline{n}}} D^{+*}_{\overline{n},r} \arrow[ur, "\rho"'] & \end{tikzcd}$$

where ψ is the map sending $e \otimes d$ to $e \otimes d$ where $e \in E_{\Delta_{n-1}}$ and $d \in D^{+*}_{\overline{n},r}$, while ρ is the map sending $e \otimes d$ to $e\varphi_i(d)$ where $e \in E^{+}_{\overline{n}}$ and $d \in D^{+*}_{\overline{n},r}$.

Since $\{1, X_i, \ldots, X_i^{q-1}\}$ is a common basis of $E^{+}_{\overline{n}}$ over $\varphi_i(E^{+}_{\overline{n}})$ and of $E_{\Delta_{n-1}}$ over $\varphi_i(E_{\Delta_{n-1}})$, it follows that both $E_{\Delta_{n-1}} \otimes_{\varphi_i, E_{\Delta_{n-1}}} D^{+*}_{\overline{n},r}$ and $E^{+}_{\overline{n}} \otimes_{\varphi_i, E^{+}_{\overline{n}}} D^{+*}_{\overline{n},r}$ can be identified with $\left(D^{+*}_{\overline{n},r}\right)^{\oplus q}$ and that ψ is a bijection. By Proposition 4.14 (ii) we know that the map

$$\begin{aligned} E^{+}_{\overline{n}} \otimes_{\varphi_i, E^{+}_{\overline{n}}} D^{+*}_{\overline{n}} &\longrightarrow D^{+*}_{\overline{n}} \\ e \otimes d &\longmapsto e\varphi_i(d) \end{aligned}$$

is bijective. Tensoring both sides on the right by $E^{+}_{\overline{n},r}$ over $E^{+}_{\overline{n}}$ we get that ρ is bijective. Therefore φ_i^{lin} is bijective as well. $\square$

Using the induction hypothesis and Lemma 4.4 (iii), it follows that the map

$$\begin{aligned} E^{\mathrm{sep}}_{\Delta_{n-1}} \otimes_{\kappa_L} \bigcap_{i \in \Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} D^{+*}_{\overline{n},r}\right)^{\varphi_i = \mathrm{id}} &\longrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} D^{+*}_{\overline{n},r} \\ e \otimes d &\longmapsto ed \end{aligned} \tag{4.11}$$

is an isomorphism for all $r > 0$. Since $D^{+*}_{\overline{n},r}$ is an $E^{+}_{\overline{n},r}$-module, by Lemma 4.16 (ii) we have an identification

$$E^{\mathrm{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} D^{+*}_{\overline{n},r} \simeq E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^{+}_{\overline{n},r}} D^{+*}_{\overline{n},r}$$

that respects the relevant group actions and Frobenius operators on both sides. By Lemma 4.15 (i)

$$\bigcap_{i \in \Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} D^{+*}_{\overline{n},r}\right)^{\varphi_i = \mathrm{id}} \simeq \bigcap_{i \in \Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^{+}_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)^{\varphi_i = \mathrm{id}}$$

is a module over $\kappa_L[X_n]/(X_n^r)$ and by Lemma 4.15 (ii) it follows that the isomorphism (4.11) can also be written as

$$\begin{aligned} &E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{\kappa_L[X_n]/(X_n^r)} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)^{\varphi_i=\mathrm{id}} \\ &\quad \xrightarrow{\sim} E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}. \end{aligned} \tag{4.12}$$

The following two lemmas establish the consequences of taking projective limits with respect to r in (4.12).

Lemma 4.21. *(i) The natural map*

$$\left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)/X_n^r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right) \to E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}$$

is an isomorphism.

(ii) The $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]$-module $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$ is X_n-adically complete.

(iii) The natural map

$$E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \longrightarrow \varprojlim_r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)$$

is an isomorphism.

Proof. (i) We have the natural isomorphisms

$$\begin{aligned} &\left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)/X_n^r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right) \\ &\simeq E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right) \\ &\simeq E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \\ &\simeq E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}. \end{aligned}$$

(ii) The module $D^{+*}_{\overline{n}}$ is finitely generated over $E^+_{\overline{n}}$ and since the latter is a Noetherian ring, we have short exact sequences of $E^+_{\overline{n}}$-modules

$$0 \longrightarrow K_1 \xrightarrow{f_1} \left(E^+_{\overline{n}}\right)^{\oplus k_1} \xrightarrow{g_1} D^{+*}_{\overline{n}} \longrightarrow 0 \tag{4.13}$$

$$0 \longrightarrow K_2 \xrightarrow{f_2} \left(E^+_{\overline{n}}\right)^{\oplus k_2} \xrightarrow{g_2} K_1 \longrightarrow 0 \tag{4.14}$$

for some $k_1, k_2 > 0$. Combining (4.13) and (4.14) it follows that the sequence

$$\left(E^+_{\overline{n}}\right)^{\oplus k_2} \xrightarrow{f_1\circ g_2} \left(E^+_{\overline{n}}\right)^{\oplus k_1} \xrightarrow{g_1} D^{+*}_{\overline{n}} \to 0 \tag{4.15}$$

is exact. Applying $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} -$ to (4.15) we obtain that

$$\left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus k_2} \xrightarrow{\mathrm{id}\otimes(f_1\circ g_2)} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus k_1} \xrightarrow{\mathrm{id}\otimes g_1} E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \to 0$$

is exact. We write $(-)_r$ for the functor $E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}} -$. Since D is an E_{Δ_n}-module, it has no X_n-torsion. Therefore $D^{+*}_{\overline{n}}$ being a submodule of D, has no X_n-torsion either. Applying $(-)_r$ for all $r > 0$ to (4.13) and using Lemma 4.18 for $M = D^{+*}_{\overline{n}}$ we obtain that the sequences

$$0 \longrightarrow (K_1)_r \xrightarrow{(f_1)_r} \left(E^+_{\overline{n}}\right)^{\oplus k_1}_r \xrightarrow{(g_1)_r} (D^{+*}_{\overline{n}})_r \longrightarrow 0$$

form a system of short exact sequences compatible with respect to taking natural projections. Since the functor $- \otimes_{E^+_{\overline{n}}} K_1$ is right exact and

$$E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_1}) \longrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_2}) \longrightarrow 0$$

is exact for all $r_1 \geq r_2 \geq 1$, it follows that the modules $\{(K_1)_r\}_{r\geq 1}$ satisfy the Mittag-Leffler property and that the sequence

$$0 \to \varprojlim_r (K_1)_r \xrightarrow{\varprojlim_r (f_1)_r} \varprojlim_r \left(E^+_{\overline{n}}\right)^{\oplus k_1}_r \xrightarrow{\varprojlim_r (g_1)_r} \varprojlim_r \left(D^{+*}_{\overline{n}}\right)_r \to 0 \tag{4.16}$$

is exact. Since K_1 can be identified with a submodule of a free module over $E^+_{\overline{n}}$ it is X_n-torsion free and Lemma 4.18 applies for $M = K_1$ as well. Applying the same argument for (4.14) we obtain that the sequence

$$0 \to \varprojlim_r (K_2)_r \xrightarrow{\varprojlim_r (f_2)_r} \varprojlim_r \left(E^+_{\overline{n}}\right)^{\oplus k_2}_r \xrightarrow{\varprojlim_r (g_2)_r} \varprojlim_r (K_1)_r \to 0 \tag{4.17}$$

is exact. Combining (4.16) and (4.17) it follows that

$$\varprojlim_r \left(E^+_{\overline{n}}\right)^{\oplus k_2}_r \xrightarrow{\varprojlim_r (f_1)_r \circ \varprojlim_r (g_2)_r} \varprojlim_r \left(E^+_{\overline{n}}\right)^{\oplus k_1}_r \xrightarrow{\varprojlim_r (g_1)_r} \varprojlim_r \left(D^{+*}_{\overline{n}}\right)_r \to 0$$

is exact. Since

$$\varprojlim_r \left(E^+_{\overline{n}}\right)^{\oplus m}_r \simeq \varprojlim_r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r)\right)^{\oplus m} \simeq \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus m}$$

for all $m \in \mathbb{N}_{>0}$ and

$$\varprojlim_r (f_1)_r \circ \varprojlim_r (g_2)_r = \mathrm{id} \otimes (f_1 \circ g_2),$$

it follows that $\varprojlim_r \left(D^{+*}_{\overline{n}}\right)_r$ and $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$ can both be identified with the cokernel of the map

$$\left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus k_2} \xrightarrow{\mathrm{id} \otimes (f_1 \circ g_2)} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus k_1},$$

thus they are isomorphic and the conclusion follows.

(iii) This follows from (i) and (ii). $\square$

Lemma 4.22. *(i) For $r \in \mathbb{N}_{\geq 1}$, the natural map*

$$\mathbb{W}(D_n)/X_n^r\mathbb{W}(D_n) \longrightarrow \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)^{\varphi_i=\mathrm{id}}$$

is an isomorphism.

(ii) For $r \in \mathbb{N}_{\geq 1}$, the natural map

$$\left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{\kappa_L[\![X_n]\!]} \mathbb{W}(D_n)\right) / X_n^r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{\kappa_L[\![X_n]\!]} \mathbb{W}(D_n)\right) \longrightarrow$$
$$E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{\kappa_L[X_n]/(X_n^r)} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)^{\varphi_i=\mathrm{id}}$$

is an isomorphism.

(iii) $\mathbb{W}(D_n)$ is a finite free module over $\kappa_L[\![X_n]\!]$.

(iv) D_n is a finite dimensional vector space over $\kappa_L(\!(X_n)\!)$.

(v) The $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]$-module

$$E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{\kappa_L[\![X_n]\!]} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}}$$

is X_n-adically complete.

(vi) The natural map

$$E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{\kappa_L[\![X_n]\!]} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}} \longrightarrow$$
$$\varprojlim_r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{\kappa_L[X_n]/(X_n^r)} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)^{\varphi_i=\mathrm{id}} \right)$$

is an isomorphism.

Proof. (i) For $r_1 \geq r_2 \geq 1$, we have a commutative diagram

$$\begin{tikzcd}
E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_1}) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \arrow[r] \arrow[d, "\simeq"] & E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_2}) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \arrow[r] \arrow[d, "\simeq"] & 0 \\
E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_1}) \otimes_{E^+_{\overline{n},r_1}} D^{+*}_{\overline{n},r_1} \arrow[r] & E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_2}) \otimes_{E^+_{\overline{n},r_2}} D^{+*}_{\overline{n},r_2} \arrow[r] & 0
\end{tikzcd}$$

whose upper row is exact and whose vertical arrows are bijections. Therefore the map

$$E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_1}) \otimes_{E^+_{\overline{n},r_1}} D^{+*}_{\overline{n},r_1} \longrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_2}) \otimes_{E^+_{\overline{n},r_2}} D^{+*}_{\overline{n},r_2}$$

is surjective. Using (4.12) we get that the natural map

$$E^{\mathrm{sep}}_{\Delta_{n-1}} \otimes_{\kappa_L} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_1}) \otimes_{E^+_{\overline{n},r_1}} D^{+*}_{\overline{n},r_1}\right)^{\varphi_i=\mathrm{id}} \longrightarrow$$
$$E^{\mathrm{sep}}_{\Delta_{n-1}} \otimes_{\kappa_L} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_2}) \otimes_{E^+_{\overline{n},r_2}} D^{+*}_{\overline{n},r_2}\right)^{\varphi_i=\mathrm{id}}$$

is onto for $r_1 \geq r_2 \geq 1$. Since $E^{\mathrm{sep}}_{\Delta_{n-1}}$ is faithfully flat over κ_L it follows that the map

$$\bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_1}) \otimes_{E^+_{\overline{n},r_1}} D^{+*}_{\overline{n},r_1}\right)^{\varphi_i=\mathrm{id}} \longrightarrow \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^{r_2}) \otimes_{E^+_{\overline{n},r_2}} D^{+*}_{\overline{n},r_2}\right)^{\varphi_i=\mathrm{id}}$$

is onto as well. Restricting the result of Lemma 4.21 (iii) to the fixed points of φ_i for $i \in \Delta_{n-1}$, it follows that

$$\begin{aligned}\bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}} &\simeq \bigcap_{i\in\Delta_{n-1}} \left(\varprojlim_r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)\right)^{\varphi_i=\mathrm{id}} \\ &\simeq \varprojlim_r \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)^{\varphi_i=\mathrm{id}}.\end{aligned} \tag{4.18}$$

By the above, the transition maps in the last projective system are surjective, therefore the natural map

$$\bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}} \longrightarrow \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)^{\varphi_i=\mathrm{id}} \tag{4.19}$$

is onto for all $r > 0$. By Lemma 4.21 (i), the kernel of the projection map

$$E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \to E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}$$

equals $X_n^r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)$. Therefore the kernel of (4.19) equals

$$X_n^r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right) \bigcap \mathbb{W}(D_n). \tag{4.20}$$

We now show that the latter equals $X_n^r\mathbb{W}(D_n)$. It is clear that $X_n^r\mathbb{W}(D_n)$ is contained in (4.20). For the other inclusion let $X_n^r z$ be an element of (4.20) for $z \in E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$. Then

$$X_n^r\varphi_i(z) = \varphi_i(X_n^r z) = X_n^r z$$

for all $i \in \Delta_{n-1}$. By Lemma 4.19 (ii) $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$ has no X_n-torsion, therefore

$$\varphi_i(z) = z$$

for all $i \in \Delta_{n-1}$. In other words, $z \in \mathbb{W}(D_n)$ as desired.

(ii) We have that

$$\begin{aligned}
&\left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{\kappa_L[\![X_n]\!]} \mathbb{W}(D_n)\right) / X_n^r \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{\kappa_L[\![X_n]\!]} \mathbb{W}(D_n)\right)\\
&\simeq E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} \mathbb{W}(D_n)\\
&\simeq E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{\kappa_L[X_n]/(X_n^r)} \mathbb{W}(D_n)/X_n^r\mathbb{W}(D_n)
\end{aligned}$$

and the conclusion follows by part (i).

(iii) By the induction hypothesis applied to $D^{+*}_{\overline{n},1} \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_{n-1}}, \Gamma_{\Delta_{n-1},L}, E_{\Delta_{n-1}})$ we know by Lemma 4.4 (i) that

$$\bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n) \otimes_{E^+_{\overline{n},1}} D^{+*}_{\overline{n},1}\right)^{\varphi_i=\mathrm{id}}$$

is a finitely generated vector space over κ_L. By (i), this means that $\mathbb{W}(D_n)/X_n\mathbb{W}(D_n)$ is finitely generated over $\kappa_L[\![X_n]\!]/(X_n)$. We also know that

$$\bigcap_{k\geq 1} X_n^k \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right) = 0 \tag{4.21}$$

because $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$ is X_n-adically complete by Lemma 4.21 (ii). Restricting (4.21) to $\mathbb{W}(D_n) \subseteq E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$, we deduce that $\mathbb{W}(D_n)$ is X_n-adically Hausdorff. The ring $\kappa_L[\![X_n]\!]$ is X_n-adically complete, therefore the hypotheses of Nakayama Lemma in the form of Lemma 1.51 are satisfied and it follows that $\mathbb{W}(D_n)$ is finitely generated over $\kappa_L[\![X_n]\!]$ as well. Also, by Lemma 4.19 (ii) $\mathbb{W}(D_n)$ is torsion free, therefore by the classification theorem of finitely generated modules over a PID, it follows that $\mathbb{W}(D_n)$ is a finitely generated free module over $\kappa_L[\![X_n]\!]$.

(iv) By Lemma 4.19 (iv) $D_n \simeq \mathbb{W}(D_n)[X_n^{-1}]$ and the result follows immediately from (iii).

(v) By the result of (iii) we know that

$$E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{\kappa_L[\![X_n]\!]} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}}$$

is a finite free module over $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!]$, which is clearly X_n-adically complete.

(vi) This follows by combining the results of (ii) and (v). $\square$

One can now show that the map (4.8) is an isomorphism.

Lemma 4.23. *The map*

$$\begin{aligned}
E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{\kappa_L(\!(X_n)\!)} D_n &\longrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D\\
e \otimes d &\longmapsto ed
\end{aligned}$$

is an isomorphism.

Proof. Varying $r > 0$ in the isomorphism (4.12) and taking projective limits, we know that the natural map

$$\varprojlim_r E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{\kappa_L[X_n]/(X_n^r)} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)^{\varphi_i=\mathrm{id}} \longrightarrow \varprojlim_r E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}$$

is an isomorphism. Using Lemma 4.21 (iii) on the right hand side and Lemma 4.22 (vi) on the left hand side, we get that the map

$$\begin{aligned} E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{\kappa_L[\![X_n]\!]} \mathbb{W}(D_n) &\longrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \\ e\otimes d &\longmapsto ed \end{aligned}$$

is an isomorphism. Inverting X_n on both sides, it follows that the map

$$\begin{aligned} E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{\kappa_L[\![X_n]\!]} \mathbb{W}(D_n) &\longrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \\ e\otimes d &\longmapsto ed \end{aligned}$$

is an isomorphism as well, where $\mathbb{W}(D_n)$ can be regarded as a subspace of $E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!)\otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$ by Lemma 4.19 (i). In the commutative diagram

$$\begin{array}{ccc} E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{\kappa_L[\![X_n]\!]} \mathbb{W}(D_n) & \longrightarrow & E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{\kappa_L(\!(X_n)\!)} \mathbb{W}(D_n)[X_n^{-1}] & \longrightarrow & E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E^+_{\overline{n}}[X_n^{-1}]} D^{+*}_{\overline{n}}[X_n^{-1}] \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{\kappa_L(\!(X_n)\!)} D_n & \longrightarrow & E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D \end{array}$$

whose horizontal arrows are the maps sending $e\otimes d$ to ed, the vertical arrows are isomorphisms by Lemma 4.19 (iv), hence the conclusion follows. □

Lemma 4.24. *D_n has the structure of an étale $(\varphi_n, \Gamma_{n,L})$-module over E_n and is a linear representation of $G_{\Delta_{n-1},L}$. All of these actions commute with each other.*

Proof. The fact that the actions exist and that they commute with each other is clear and by Lemma 4.22 (iv) D_n is a finite dimensional vector space over E_n. It remains to show that D_n is étale. Since $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ the map

$$\begin{aligned} \varphi_n^{\mathrm{lin}} : E_n \otimes_{\varphi_n, E_n} D &\longrightarrow D \\ e\otimes d &\longmapsto e\varphi_n(d) \end{aligned}$$

is an isomorphism. This is because both E_n and E_{Δ_n} are free modules over $\varphi_n(E_n)$ and $\varphi_n(E_{\Delta_n})$, respectively, with identical basis elements $\{X_n^s\}_{0\le s\le q-1}$. In other words, we have a commutative diagram

$$\begin{array}{ccc} E_{\Delta_n} \otimes_{\varphi_n, E_{\Delta_n}} D & \xrightarrow{\varphi_n^{\mathrm{lin}}} & D \\ \big\downarrow & & \big\| \\ E_n \otimes_{\varphi_n, E_n} D & \xrightarrow{\varphi_n^{\mathrm{lin}}} & D \end{array}$$

where the left vertical arrow is an additive bijection whose construction is analogous to that of the left vertical arrow in the diagram (4.7). The bijectivity of the top horizontal arrow forces then the bijectivity of bottom one. Also, since E_n is free over $\varphi_n(E_n)$, the functor $E_n \otimes_{\varphi_n, E_n} -$ commutes with intersections, therefore

$$\begin{aligned} E_n \otimes_{\varphi_n, E_n} D_n &\simeq E_n \otimes_{\varphi_n, E_n} \bigcap_{j \in \Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D \right)^{\varphi_j = \mathrm{id}} \\ &\simeq \bigcap_{j \in \Delta_{n-1}} \left(E_n \otimes_{\varphi_n, E_n} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D \right) \right)^{\varphi_j = \mathrm{id}} \\ &\simeq \bigcap_{j \in \Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} (E_n \otimes_{\varphi_n, E_n} D) \right)^{\varphi_j = \mathrm{id}} \\ &\simeq \bigcap_{j \in \Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D \right)^{\varphi_j = \mathrm{id}} = D_n \end{aligned}$$

and we are done. □

4.4 The embedding $D_n \subseteq E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X^{-1}_{\Delta_n}] \otimes_{E_{\Delta_n}} D$

Before we prove the essential surjectivity of $\mathbb{D}$, we need one more technical result which states that the map

$$\begin{gathered} E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X^{-1}_{\Delta_n}] \otimes_{E_{\Delta_n}} D \to E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D \\ e \otimes d \mapsto e \otimes d \end{gathered}$$

is an embedding whose image contains D_n. For that we introduce an $E^+_{\Delta_n}$-submodule of $D^{+*}_{\overline{n}}$ whose image is under good control after we apply

$$\varphi_{\overline{n}} \coloneqq \prod_{j \in \Delta_{n-1}} \varphi_j \in \mathcal{T}_{+, \Delta_n, L}.$$

Lemma 4.25. *For a large enough $k \in \mathbb{N}$, the module*

$$M_n \coloneqq X^{-k}_{\Delta_{n-1}} \left(D^+ \cap D^{+*}_{\overline{n}} \right)$$

has the properties:

(i) M_n *is a finitely generated* $E^+_{\Delta_n}$*-submodule of* $D^{+*}_{\overline{n}}$*,*

(ii) $D^{+*}_{\overline{n}} = M_n[X^{-1}_{\Delta_{n-1}}]$ *and*

(iii) $D^{+*}_{\overline{n}} = \bigcup_{l\geq 0} E^+_{\Delta_n}\varphi^l_{\overline{n}}(M_n)$*.*

Proof. We first observe that for $D_0 \coloneqq D^+ \cap D^{+*}_{\overline{n}}$, we have the identity

$$D_0[X^{-1}_{\Delta_{n-1}}] = D^{+*}_{\overline{n}}. \tag{4.22}$$

The inclusion $D_0[X^{-1}_{\Delta_{n-1}}] \subseteq D^{+*}_{\overline{n}}$ is clear because $D_0 \subseteq D^{+*}_{\overline{n}}$. For the reverse inclusion, consider $y \in D^{+*}_{\overline{n}}$. Then $y \in D^+_{\overline{n}} = D^+[X^{-1}_{\Delta_{n-1}}]$, and therefore $X^m_{\Delta_{n-1}} y \in D^+$ for some $m \geq 0$. It is also clear that $X^m_{\Delta_{n-1}} y \in D^{+*}_{\overline{n}}$ as well. In other words $X^m_{\Delta_{n-1}} y \in D_0$ and thus $y \in D_0[X^{-1}_{\Delta_{n-1}}]$ as desired.

By Proposition 4.14 (ii) applied for $\tau = \varphi_{\overline{n}}$, we know that

$$D_0[X^{-1}_{\Delta_{n-1}}] = D^{+*}_{\overline{n}} = E^+_{\overline{n}}\varphi_{\overline{n}}(D^{+*}_{\overline{n}}) = E^+_{\overline{n}}\varphi_{\overline{n}}(D_0[X^{-1}_{\Delta_{n-1}}]) = E^+_{\overline{n}}\varphi_{\overline{n}}(D_0).$$

By Proposition 4.9 (iii) and the Noetherianity of $E^+_{\Delta_n}$, we know that D_0 is a finitely generated $E^+_{\Delta_n}$-submodule of D^+. Therefore

$$X^{k_0}_{\Delta_{n-1}} D_0 \subseteq E^+_{\Delta_n}\varphi_{\overline{n}}(D_0)$$

for some $k_0 \in \mathbb{N}_{>0}$ large enough. For $k_1 > \frac{k_0}{q-1}$, we then have that

$$\begin{aligned} X^{-k_1}_{\Delta_{n-1}} D_0 &\subseteq X^{-k_1-k_0}_{\Delta_{n-1}} E^+_{\Delta_n}\varphi_{\overline{n}}(D_0) \\ &\subseteq X^{-qk_1}_{\Delta_{n-1}} E^+_{\Delta_n}\varphi_{\overline{n}}(D_0) \\ &= E^+_{\Delta_n}\varphi_{\overline{n}}(X^{-k_1}_{\Delta_{n-1}} D_0). \end{aligned} \tag{4.23}$$

Set $M \coloneqq X^{-k_1}_{\Delta_{n-1}} D_0$ and $M_n \coloneqq X^{-1}_{\Delta_{n-1}} M$. We know that M_n satisfies (i) because D_0 is finitely generated over $E^+_{\Delta_n}$ and by (4.22) we also know that M_n satifies (ii). By (4.23), we have that $M \subseteq E^+_{\Delta_n}\varphi_{\overline{n}}(M)$. Iterating the latter, we obtain that

$$M \subseteq E^+_{\Delta_n}\varphi^l_{\overline{n}}(M)$$

for all $l \in \mathbb{N}$. Therefore

$$\begin{aligned} X^{-q^l+1}_{\Delta_{n-1}} M_n &= X^{-q^l}_{\Delta_{n-1}} M \\ &\subseteq X^{-q^l}_{\Delta_{n-1}} E^+_{\Delta_n}\varphi^l_{\overline{n}}(M) \\ &= E^+_{\Delta_n}\varphi^l_{\overline{n}}(X^{-1}_{\Delta_{n-1}} M) \\ &= E^+_{\Delta_n}\varphi^l_{\overline{n}}(M_n). \end{aligned}$$

Then M_n satisfies property (iii) because $D^{+*}_{\overline{n}} = \bigcup_{l\geq 0} X^{-q^l+1}_{\Delta_{n-1}} M_n$ by (ii). $\square$

We fix a $k \in \mathbb{N}$ for which the conditions of Lemma 4.25 are satisfied, as well as the corresponding module M_n. We now state the following counterpart of Lemma 4.18 for $E^+_{\Delta_n}$-modules, whose proof is analogous, but for completeness we provide the details here.

Lemma 4.26. *Let M be an $E^+_{\Delta_n}$-module with no X_n-torsion. Then for all $i > 0$*

(i) $\mathrm{Tor}_i^{E^+_{\Delta_n}}\left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r), M\right) = 0,$

(ii) $\mathrm{Tor}_i^{E^+_{\Delta_n}}\left(E'^+_{\Delta_{n-1}}[X_n]/(X_n^r), M\right) = 0$ *for any collection of finite separable extensions $(E'_j/E_j)_{j\in\Delta_{n-1}}$.*

Proof. Since M has no X_n-torsion, it is torsion free hence flat over $\kappa_L[\![X_n]\!]$, because $\kappa_L[\![X_n]\!]$ is a PID. Therefore

$$\mathrm{Tor}_i^{\kappa_L[\![X_n]\!]}\left(\kappa_L[X_n]/(X_n^r), M\right) = 0 \tag{4.24}$$

for all $i > 0$. Let

$$P_\bullet \longrightarrow M \longrightarrow 0 \tag{4.25}$$

be a projective resolution of M in $E^+_{\Delta_n} - \mathrm{Mod}$. Applying $E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} -$ to (4.25), the i-th homology group of

$$E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} P_\bullet \longrightarrow 0$$

computes $\mathrm{Tor}_i^{E^+_{\Delta_n}}\left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r), M\right)$ for all $i > 0$. To prove (i), by Lemma 4.16 (iv) it suffices to show that the complex

$$\begin{aligned} &E^{\mathrm{sep}\,+}_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} P_\bullet\right) \longrightarrow \\ &\longrightarrow E^{\mathrm{sep}\,+}_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} M\right) \longrightarrow 0 \end{aligned} \tag{4.26}$$

is exact. Since P_i is projective over $E^+_{\Delta_n}$, it is then torsion free over $E^+_{\Delta_n}$, thus torsion free, hence flat over $\kappa_L[\![X_n]\!]$. This means that (4.25) is a resolution of flat modules in $\kappa_L[\![X_n]\!] - \mathrm{Mod}$. Since flat resolutions also allow us to compute the Tor functors, by (4.24) it follows that

$$\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} P_\bullet \longrightarrow \kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} M \longrightarrow 0 \tag{4.27}$$

is an exact complex. By Lemma 3.12 (ii) we know that $E^{\mathrm{sep}\,+}_{\Delta_{n-1}}$ is a flat $E^+_{\Delta_{n-1}}$-module, thus (4.26) is exact, which shows that (i) holds.

Applying $E'^+_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} -$ to (4.25) instead, the i-th homology group of

$$E'^+_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} P_\bullet \longrightarrow 0$$

computes $\mathrm{Tor}_i^{E^+_{\Delta_n}}\left(E'^+_{\Delta_{n-1}}[X_n]/(X_n^r), M\right)$ for all $i > 0$. To prove (ii), by Lemma 4.17 (i) it suffices to show that the complex

$$\begin{aligned}E'^+_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} P_\bullet\right) \longrightarrow \\ \longrightarrow E'^+_{\Delta_{n-1}} \otimes_{E^+_{\Delta_{n-1}}} \left(\kappa_L[X_n]/(X_n^r) \otimes_{\kappa_L[\![X_n]\!]} M\right) \longrightarrow 0\end{aligned}$$

is exact. Combining the exactness of (4.27) with the fact that $E'^+_{\Delta_{n-1}}$ is a flat $E^+_{\Delta_{n-1}}$-module, the conclusion follows. □

Lemma 4.27. *Let $r \in \mathbb{N}_{>0}$.*

(i) The map

$$E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n \longrightarrow E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}}$$

induced by the inclusion $M_n \subseteq D^{+*}_{\overline{n}}$ is injective.

(ii) For a collection of finite separable extensions $(E'_j/E_j)_{j\in\Delta_{n-1}}$, the map

$$E'^+_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n \longrightarrow E'^+_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}}$$

induced by the inclusion $M_n \subseteq D^{+*}_{\overline{n}}$ is injective.

Proof. By Lemma 4.25 we know that $M_n = X^{-k}_{\Delta_{n-1}}\left(D^+ \cap D^{+*}_{\overline{n}}\right)$ for some $k > 0$. Therefore

$$\begin{aligned}D^{+*}_{\overline{n}}/M_n &= D^{+*}_{\overline{n}}/X^{-k}_{\Delta_{n-1}}\left(D^+ \cap D^{+*}_{\overline{n}}\right)\\ &= D^{+*}_{\overline{n}}/\left(X^{-k}_{\Delta_{n-1}}D^+ \cap D^{+*}_{\overline{n}}\right)\\ &\simeq \left(D^{+*}_{\overline{n}} + X^{-k}_{\Delta_{n-1}}D^+\right)/X^{-k}_{\Delta_{n-1}}D^+\end{aligned}$$

is X_n-torsion free as the latter is contained in $D^+_{\overline{n}}/\left(X^{-k}_{\Delta_{n-1}}D^+\right) \simeq D^+_{\overline{n}}/D^+$, which has no X_n-torsion by Lemma 4.11. Applying Lemma 4.26 for $M = D^{+*}_{\overline{n}}/M_n$, the conclusion follows. □

Lemma 4.28. *Let $r \in \mathbb{N}_{>0}$ and $(E'_j/E_j)_{j\in\Delta_{n-1}}$ be a collection of finite separable extensions.*

(i) The map

$$E'^+_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}} \longrightarrow E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}}$$

induced by the inclusion $E'^+_{\Delta_{n-1}}[X_n]/(X_n^r) \to E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r)$ is injective.

(ii) The map

$$E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n \longrightarrow E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n$$

induced by the inclusion $E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r) \to E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r)$ *is injective.*

Proof. (i) Consider the commutative diagram

$$\begin{tikzcd}
E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}} \arrow[r] \arrow[d, "\simeq"] & E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}} \arrow[d, "\simeq"] \\
E'_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \arrow[r] \arrow[d, "\simeq"] & E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \arrow[d, "\simeq"] \\
E'_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r} \arrow[r] \arrow[d, "\simeq"] & E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r} \arrow[d, "\simeq"] \\
E'_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} D^{+*}_{\overline{n},r} \arrow[r] & E^{\mathrm{sep}}_{\Delta_{n-1}} \otimes_{E_{\Delta_{n-1}}} D^{+*}_{\overline{n},r}
\end{tikzcd}$$

whose two middle horizontal arrows are the natural maps induced by the embedding

$$E'_{\Delta_{n-1}}[X_n]/(X_n^r) \hookrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r)$$

and whose bottom horizontal arrow is the natural map induced by the embedding

$$E'_{\Delta_{n-1}} \hookrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}}.$$

Its bottom left vertical arrows is an isomorphism by Lemma 4.17 (ii), while its bottom right vertical arrows is an isomorphism by Lemma 4.16 (iii). The other vertical arrows are clearly isomorphisms as well. The bottom horizontal arrow is an embedding because our induction hypothesis and Lemma 4.4 (i) applied for

$$D^{+*}_{\overline{n},r} \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_{n-1}}, \Gamma_{\Delta_{n-1},L}, E_{\Delta_{n-1}})$$

shows that $D^{+*}_{\overline{n},r}$ is a flat $E_{\Delta_{n-1}}$-module. Therefore the top horizontal arrow is an embedding, as desired.

(ii) Consider the commutative diagram

$$\begin{tikzcd}
E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n \arrow[r] \arrow[d] & E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n \arrow[d] \\
E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}} \arrow[r] & E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}}
\end{tikzcd}$$

whose horizontal arrows are induced by the embedding

$$E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r) \to E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r)$$

and whose vertical arrows are induced by the inclusion $M_n \subseteq D_{\overline{n}}^{+*}$. Its left vertical arrow is injective by Lemma 4.27 (ii) and its bottom horizontal arrow is injective by part (i). Therefore its top horizontal arrow is also injective, as desired. $\square$

Lemma 4.29. *For all $r > 0$,*

$$\bigcap_{i\in\Delta_{n-1}} \left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[X_n]/(X_n^r) \otimes_{E_{\Delta_n}^+} D_{\overline{n}}^{+*}\right)^{\varphi_i=\mathrm{id}} \tag{4.28}$$

is contained in $E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[X_n]/(X_n^r) \otimes_{E_{\Delta_n}^+} M_n$.

Proof. By the induction hypothesis and Lemma 4.4 (ii) applied for $D_{\overline{n},r}^{+*}$ we know that

$$\begin{aligned}\bigcap_{i\in\Delta_{n-1}} \left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[X_n]/(X_n^r) \otimes_{E_{\Delta_n}^+} D_{\overline{n}}^{+*}\right)^{\varphi_i=\mathrm{id}} &\simeq \bigcap_{i\in\Delta_{n-1}} \left(E_{\Delta_{n-1}}^{\mathrm{sep}}[X_n]/(X_n^r) \otimes_{E_{\overline{n}}^+} D_{\overline{n}}^{+*}\right)^{\varphi_i=\mathrm{id}} \\ &\simeq \bigcap_{i\in\Delta_{n-1}} \left(E_{\Delta_{n-1}}^{\mathrm{sep}}[X_n]/(X_n^r) \otimes_{E_{\overline{n},r}^+} D_{\overline{n},r}^{+*}\right)^{\varphi_i=\mathrm{id}}\end{aligned}$$

is a finitely generated κ_L-vector space, in particular it is a finite set. Recall that by Lemma 4.25 we have that $D_{\overline{n}}^{+*} = \bigcup_{l\geq 0} E_{\Delta_n}^+ \varphi_{\overline{n}}^l(M_n)$. Hence for a large enough $\ell \geq 0$, (4.28) is contained in

$$E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[X_n]/(X_n^r)\varphi_{\overline{n}}^{\ell}\left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[X_n]/(X_n^r) \otimes_{E_{\Delta_n}^+} M_n\right).$$

Let x be an element of (4.28). Because

$$x \in E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[X_n]/(X_n^r)\varphi_{\overline{n}}^{\ell}\left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[X_n]/(X_n^r) \otimes_{E_{\Delta_n}^+} M_n\right),$$

we can write

$$x = \sum_{j=1}^{s} e_j \varphi_{\overline{n}}^{\ell}(y_j)$$

for some $e_j \in E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[X_n]/(X_n^r)$ and $y_j \in E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[X_n]/(X_n^r) \otimes_{E_{\Delta_n}^+} M_n$. Using the colimit description of $E_{\Delta_{n-1}}^{\mathrm{sep}\,+}$ and Lemma 4.28, we can consider a collection of finite separable extensions $(E'_j/E_j)_{j\in\Delta_{n-1}}$ for which

$$x \in E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E_{\Delta_n}^+} D_{\overline{n}}^{+*} \simeq E'_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E_{\overline{n}}^+} D_{\overline{n}}^{+*},$$

while $e_j \in E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r)$ and $y_j \in E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E_{\Delta_n}^+} M_n$ for every $1 \leq j \leq s$. To simplify notation, we let

$$E'_{\Delta_{n-1},r} \coloneqq E'_{\Delta_{n-1}}[X_n]/(X_n^r)$$

and

$$E'^{+}_{\Delta_{n-1},r} \coloneqq E'^{+}_{\Delta_{n-1}}[X_n]/(X_n^r).$$

Consider the map

$$\left(\varphi_{\overline{n}}^{\ell}\right)^{\mathrm{lin}}: \quad E'_{\Delta_{n-1},r} \otimes_{E'_{\Delta_{n-1},r},\varphi_{\overline{n}}^{\ell}} \left(E'_{\Delta_{n-1},r} \otimes_{E_{\overline{n}}^{+}} D_{\overline{n}}^{+*}\right) \longrightarrow E'_{\Delta_{n-1},r} \otimes_{E_{\overline{n}}^{+}} D_{\overline{n}}^{+*}$$

$$e \otimes (f \otimes d) \longmapsto e\varphi_{\overline{n}}^{\ell}(f) \otimes \varphi_{\overline{n}}^{\ell}(d).$$

We claim it is bijective. Indeed, we can factor it through the maps

$$\psi : E'_{\Delta_{n-1},r} \otimes_{E'_{\Delta_{n-1},r},\varphi_{\overline{n}}^{\ell}} \left(E'_{\Delta_{n-1},r} \otimes_{E_{\overline{n}}^{+}} D_{\overline{n}}^{+*}\right) \to \left(E'_{\Delta_{n-1},r} \otimes_{E'_{\Delta_{n-1},r},\varphi_{\overline{n}}^{\ell}} E'_{\Delta_{n-1},r}\right) \otimes_{E_{\overline{n}}^{+}} \left(E_{\overline{n}}^{+} \otimes_{E_{\overline{n}}^{+},\varphi_{\overline{n}}^{\ell}} D_{\overline{n}}^{+*}\right)$$

and

$$\psi' : \left(E'_{\Delta_{n-1},r} \otimes_{E'_{\Delta_{n-1},r},\varphi_{\overline{n}}^{\ell}} E'_{\Delta_{n-1},r}\right) \otimes_{E_{\overline{n}}^{+}} \left(E_{\overline{n}}^{+} \otimes_{E_{\overline{n}}^{+},\varphi_{\overline{n}}^{\ell}} D_{\overline{n}}^{+*}\right) \longrightarrow E'_{\Delta_{n-1},r} \otimes_{E_{\overline{n}}^{+}} D_{\overline{n}}^{+*}.$$

The map ψ is defined by the formula $\psi\left(e \otimes (f \otimes d)\right) = (e \otimes f) \otimes (1 \otimes d)$ and has as inverse the map sending $(e \otimes f) \otimes (c \otimes d)$ to $ce \otimes (f \otimes d)$ and thus is bijective. The map ψ' is defined to be the tensor product of the linearizations of $\varphi_{\overline{n}}^{\ell}$ on $E'_{\Delta_{n-1},r}$ and $D_{\overline{n}}^{+*}$. The linearization of $\varphi_{\overline{n}}^{\ell}$ on $E'_{\Delta_{n-1},r}$ is clearly bijective and the other one is bijective by Proposition 4.14 (ii). Therefore $(\varphi_{\overline{n}}^{\ell})^{\mathrm{lin}} = \psi' \circ \psi$ is bijective as well. Note that

$$(\varphi_{\overline{n}}^{\ell})^{\mathrm{lin}}\left(\sum_{j=1}^{s} e_j \otimes y_j\right) = \sum_{j=1}^{s} e_i\varphi_{\overline{n}}^{\ell}(y_j) = x = \varphi_{\overline{n}}^{\ell}(x) = (\varphi_{\overline{n}}^{\ell})^{\mathrm{lin}}(1 \otimes x).$$

Therefore $1 \otimes x = \sum_{j=1}^{s} e_j \otimes y_j$ as elements of $E'_{\Delta_{n-1},r} \otimes_{E'_{\Delta_{n-1},r},\varphi_{\overline{n}}^{\ell}} \left(E'_{\Delta_{n-1},r} \otimes_{E_{\overline{n}}^{+}} D_{\overline{n}}^{+*}\right)$. We now show that this implies that

$$x \in E'^{+}_{\Delta_{n-1},r} \otimes_{E_{\Delta_n}^{+}} M_n \subseteq E'_{\Delta_{n-1},r} \otimes_{E_{\overline{n}}^{+}} D_{\overline{n}}^{+*}.$$

For each $i \in \Delta_{n-1}$, write $E'_i = \mathbb{F}_{q_i}(\!(X'_i)\!)$. This identifies $E'_{\Delta_{n-1}}$ with

$$\left(\bigotimes_{i \in \Delta_{n-1},\kappa_L} \mathbb{F}_{q_i}\right) [\![X'_i : i \in \Delta_{n-1}]\!][X_{\Delta_{n-1}}^{-1}]$$

and $\varphi_{\overline{n}}^{\ell}$ with the $q^{\ell}-$th power map. The ring $\bigotimes_{i \in \Delta_{n-1},\kappa_L} \mathbb{F}_{q_i}$ is a finite dimensional étale algebra over κ_L, being a tensor product of finite dimensional étale algebras over κ_L. Therefore it is reduced and $\varphi_{\overline{n}}^{\ell}$ is an injective κ_L-linear endomorphism of $\bigotimes_{i \in \Delta_{n-1},\kappa_L} \mathbb{F}_{q_i}$. By dimension reasons it also follows that $\varphi_{\overline{n}}^{\ell}$ is bijective on $\bigotimes_{i \in \Delta_{n-1},\kappa_L} \mathbb{F}_{q_i}$. Then $E'_{\Delta_{n-1}}$ is a free module over $\varphi_{\overline{n}}^{\ell}\left(E'_{\Delta_{n-1}}\right)$ whose basis

$$\left\{\prod_{i \in \Delta_{n-1}} X_i'^{s_i} : 0 \leq s_i \leq q^{\ell} - 1\right\}$$

contains the element 1. Therefore

$$\mathfrak{B} := \left\{ X_n^s \prod_{i\in\Delta_{n-1}} X_i'^{s_i} : 0 \leq s \leq r-1, 0 \leq s_i \leq q^\ell - 1 \right\}$$

is a basis of $E'_{\Delta_{n-1},r}$ over $\varphi_{\overline{n}}^{\ell}\left(E'_{\Delta_{n-1},r}\right)$ containing 1. Using this basis we have an isomorphism

$$\rho : E'_{\Delta_{n-1},r} \otimes_{E'_{\Delta_{n-1},r},\varphi_{\overline{n}}^{\ell}} \left(E'_{\Delta_{n-1},r} \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right) \to \left(E'_{\Delta_{n-1},r} \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\oplus\mathfrak{B}}$$

of $E'_{\Delta_{n-1},r}$-modules. Since $1 \in \mathfrak{B}$, we know that $\rho(1\otimes x) = (x_\beta)_{\beta\in\mathfrak{B}}$ where $x_1 = x$ and $x_\beta = 0$ for every $\beta \in \mathfrak{B} - \{1\}$. On the other hand, $\mathfrak{B}$ is also a basis of $E'^+_{\Delta_{n-1},r}$ over $\varphi_{\overline{n}}^{\ell}\left(E'^+_{\Delta_{n-1},r}\right)$, therefore

$$\rho\left(\sum_{j=1}^{s} e_j \otimes y_j\right) \in \left(E'^+_{\Delta_{n-1},r} \otimes_{E^+_{\Delta_n}} M_n\right)^{\oplus\mathfrak{B}}$$

because $e_j \in E'^+_{\Delta_{n-1},r}$ and $y_j \in E'^+_{\Delta_{n-1},r} \otimes_{E^+_{\Delta_n}} M_n$. Therefore

$$\rho(1\otimes x) \in \left(E'^+_{\Delta_{n-1},r} \otimes_{E^+_{\Delta_n}} M_n\right)^{\oplus\mathfrak{B}},$$

which means that $x \in E'^+_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n \subseteq E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n$, as desired. $\square$

The following analog of Lemma 4.21 holds for M_n as well.

Lemma 4.30. *The natural map*

$$E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\Delta_n}} M_n \longrightarrow \varprojlim_r \left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n\right)$$

is an isomorphism.

Proof. The right hand side is the X_n-adic completion of the left hand side, hence we need to show that $E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\Delta_n}} M_n$ is X_n-adically complete. The module M_n is finitely generated over $E^+_{\Delta_n}$ and since the latter is a Noetherian ring, we have the short exact sequences

$$0 \longrightarrow K_1 \xrightarrow{f_1} \left(E^+_{\Delta_n}\right)^{\oplus\ell_1} \xrightarrow{g_1} M_n \longrightarrow 0 \tag{4.29}$$

$$0 \longrightarrow K_2 \xrightarrow{f_2} \left(E^+_{\Delta_n}\right)^{\oplus\ell_2} \xrightarrow{g_2} K_1 \longrightarrow 0 \tag{4.30}$$

for some $\ell_1, \ell_2 > 0$. Combining (4.29) and (4.30) it follows that

$$\left(E^+_{\Delta_n}\right)^{\oplus\ell_2} \xrightarrow{f_1\circ g_2} \left(E^+_{\Delta_n}\right)^{\oplus\ell_1} \xrightarrow{g_1} M_n \to 0$$

is exact. Applying $E^{\mathrm{sep}+}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\Delta_n}} -$ we obtain that

$$\left(E^{\mathrm{sep}+}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus \ell_2} \xrightarrow{\mathrm{id}\otimes(f_1\circ g_2)} \left(E^{\mathrm{sep}+}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus \ell_1} \xrightarrow{\mathrm{id}\otimes g_1} E^{\mathrm{sep}+}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\Delta_n}} M_n \to 0$$

is exact. We write $(-)_r$ for the functor $E^{\mathrm{sep}+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} -$. Since $M_n \subseteq D$, it has no X_n-torsion. Applying $(-)_r$ to (4.29) and using Lemma 4.26 (i) for $M = M_n$ we obtain that the sequences

$$0 \longrightarrow (K_1)_r \xrightarrow{(f_1)_r} \left(E^+_{\Delta_n}\right)^{\oplus \ell_1}_r \xrightarrow{(g_1)_r} (M_n)_r \longrightarrow 0$$

are exact and form a system of compatible sequences with respect to r. Since the functor $K_1 \otimes_{E^+_{\Delta_n}} -$ is right exact and

$$E^{\mathrm{sep}+}_{\Delta_{n-1}}[X_n]/(X_n^{r_1}) \longrightarrow E^{\mathrm{sep}+}_{\Delta_{n-1}}[X_n]/(X_n^{r_2}) \longrightarrow 0$$

is exact for all $r_1 \geq r_2 \geq 1$, it follows that the modules $\{(K_1)_r\}_{r\geq 1}$ satisfy the Mittag-Leffler property and

$$0 \to \varprojlim_r (K_1)_r \xrightarrow{\varprojlim_r (f_1)_r} \varprojlim_r \left(E^+_{\Delta_n}\right)^{\oplus \ell_1}_r \xrightarrow{\varprojlim_r (g_1)_r} \varprojlim_r (M_n)_r \to 0 \tag{4.31}$$

is exact. Since K_1 can be identified with a submodule of a free module over $E^+_{\Delta_n}$ it is X_n-torsion free and Lemma 4.26 (i) applies for $M = K_1$ as well. Using the same argument for (4.30) we obtain that

$$0 \to \varprojlim_r (K_2)_r \xrightarrow{\varprojlim_r (f_2)_r} \varprojlim_r \left(E^+_{\Delta_n}\right)^{\oplus \ell_2}_r \xrightarrow{\varprojlim_r (g_2)_r} \varprojlim_r (K_1)_r \to 0 \tag{4.32}$$

is exact. Combining (4.31) and (4.32) it follows that

$$\varprojlim_r \left(E^+_{\Delta_{n-1}}\right)^{\oplus \ell_2}_r \xrightarrow{\varprojlim_r (f_1)_r \circ \varprojlim_r (g_2)_r} \varprojlim_r \left(E^+_{\Delta_{n-1}}\right)^{\oplus \ell_1}_r \xrightarrow{\varprojlim_r (g_1)_r} \varprojlim_r (M_n)_r \to 0$$

is exact. Since $\varprojlim_r \left(E^+_{\Delta_{n-1}}\right)^{\oplus m}_r \simeq \left(E^{\mathrm{sep}+}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus m}$ for all $m \in \mathbb{N}_{>0}$ and

$$\varprojlim_r (f_1)_r \circ \varprojlim_r (g_2)_r = \mathrm{id} \otimes (f_1 \circ g_2)$$

it follows that both $\varprojlim_r (M_n)_r$ and $E^{\mathrm{sep}+}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\Delta_n}} M_n$ can be identified with the cokernel of the map

$$\left(E^{\mathrm{sep}+}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus \ell_2} \xrightarrow{\mathrm{id}\otimes(f_1\circ g_2)} \left(E^{\mathrm{sep}+}_{\Delta_{n-1}}[\![X_n]\!]\right)^{\oplus \ell_1}$$

thus they are isomorphic and the conclusion follows. □

We can prove now the claim made at the beginning of the section.

Proposition 4.31. *The map*

$$E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D \to E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D$$

induced by the inclusion $E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}] \subseteq E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!)$ *is an embedding and its image contains* D_n*.*

Proof. By Lemma 4.27 (i) and Lemma 4.29 we know that the natural map

$$E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} M_n \hookrightarrow E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}} \tag{4.33}$$

induced by the inclusion $M_n \subseteq D^{+*}_{\overline{n}}$ is a monomorphism whose image contains

$$\bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}}.$$

We have the isomorphisms

$$\begin{aligned} E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}} &\simeq E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \\ &\simeq E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r} \end{aligned}$$

and

$$\begin{aligned} \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\Delta_n}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}} &\simeq \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}} \\ &\simeq \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[X_n]/(X_n^r) \otimes_{E^+_{\overline{n},r}} D^{+*}_{\overline{n},r}\right)^{\varphi_i=\mathrm{id}}. \end{aligned}$$

Taking projective limits in (4.33) and using the results of Lemmas 4.21, 4.22 and 4.30, we obtain that the map

$$\begin{gathered} E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\Delta_n}} M_n \hookrightarrow E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \\ e\otimes d \mapsto e \otimes d \end{gathered} \tag{4.34}$$

is a monomorphism whose image contains

$$\bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}}.$$

Note that $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}$ has no X_n-torsion by Lemma 4.19 (i). Moreover, $X_{\Delta_{n-1}}$ is a unit in $E^+_{\overline{n}}$, therefore the natural maps

$$\begin{aligned} E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} &\longrightarrow \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)[X_n^{-1}] \\ &\longrightarrow \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)[X_{\Delta_n}^{-1}] \end{aligned}$$

are injective. We then know that $\bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}} [X_n^{-1}]$ is identified with a subspace of $\left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)[X_{\Delta_n}^{-1}]$.

Invert X_{Δ_n} in (4.34) and consider the commutative diagram

$$\begin{tikzcd}
\left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\Delta_n}} M_n\right)[X_{\Delta_n}^{-1}] \arrow[r] \arrow[d, "\simeq"] & \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)[X_{\Delta_n}^{-1}] \arrow[d, "\simeq"] \\
E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E^+_{\Delta_n}} M_n \arrow[r] \arrow[d, "\simeq"] & E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}} \arrow[d, "\simeq"] \\
E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D \arrow[r] & E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D
\end{tikzcd}$$

whose arrows are the natural maps. The bottom left vertical arrow is an isomorphism because

$$M_n[X_{\Delta_n}^{-1}] \simeq M_n[X_{\Delta_{n-1}}^{-1}][X_n^{-1}] \simeq D^{+*}_{\overline{n}}[X_n^{-1}] \simeq D$$

by Lemma 4.25. The bottom right vertical arrow is an isomorphism because $D^{+*}_{\overline{n}}[X_n^{-1}] \simeq D$ and $E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}] \simeq E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!)$. The top horizontal arrow is an embedding because localization is exact, therefore the bottom horizontal arrow is an embedding whose image contains

$$\begin{aligned}
\bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}} [X_n^{-1}] &\simeq \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}[\![X_n]\!][X_n^{-1}] \otimes_{E^+_{\overline{n}}} D^{+*}_{\overline{n}}\right)^{\varphi_i=\mathrm{id}} \\
&\simeq \bigcap_{i\in\Delta_{n-1}} \left(E^{\mathrm{sep}}_{\Delta_{n-1}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D\right)^{\varphi_i=\mathrm{id}}
\end{aligned}$$

where the last isomorphism holds because φ_i acts trivially on X_n for $i \in \Delta_{n-1}$. □

4.5 The essential surjectivity of $\mathbb{D}$

Lemma 4.32. *The group $H_{\Delta_{n-1},L}$ acts on $E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}]$ and*

$$\left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}]\right)^{H_{\Delta_{n-1},L}} = E_{\Delta_n}.$$

Proof. Applying Lemma 3.22 for the set Δ_{n-1} shows that $H_{\Delta_{n-1},L}$ acts on $E^{\mathrm{sep}\,+}_{\Delta_{n-1}}$, hence $H_{\Delta_{n-1},L}$ acts on $E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}]$ as well. By Corollary 3.41 (ii) applied for the set Δ_{n-1}, we also know that

$$\left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}\right)^{H_{\Delta_{n-1},L}} = E^+_{\Delta_{n-1}}.$$

Therefore

$$\left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!]\right)^{H_{\Delta_{n-1},L}} = E^{+}_{\Delta_{n-1}}[\![X_n]\!] = E^{+}_{\Delta_n}.$$

Since every element of $H_{\Delta_{n-1},L}$ fixes X_{Δ_n}, we have that

$$\left(E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}]\right)^{H_{\Delta_{n-1},L}} = E^{+}_{\Delta_n}[X_{\Delta_n}^{-1}] = E_{\Delta_n}$$

as desired. $\square$

We also make use of the following generalization of the embedding in (3.10).

Lemma 4.33. *Let R be an integral domain and A and B be commutative flat algebras over R. Then the natural map*

$$\psi : A \otimes_R B[\![X]\!] \to (A \otimes_R B)[\![X]\!]$$

defined by $\psi\left(\sum_{i=1}^{m} a_i \otimes \left(\sum_{j=0}^{\infty} b_{ij}X^j\right)\right) = \sum_{j=0}^{\infty}\left(\sum_{i=1}^{m} a_i \otimes b_{ij}\right) X^j$ *is injective.*

Proof. Let $Q = \mathrm{Frac}(R)$ and $(-)_Q \coloneqq - \otimes_R Q$. Consider the diagram

$$\begin{array}{ccc} A \otimes_R B[\![X]\!] & \longrightarrow & A_Q \otimes_Q B_Q[\![X]\!] \\ \big\downarrow{\scriptstyle\psi} & & \big\downarrow{\scriptstyle\psi_Q} \\ (A \otimes_R B)[\![X]\!] & \longrightarrow & (A_Q \otimes_Q B_Q)[\![X]\!] \end{array} \tag{4.35}$$

where the horizontal arrows are the natural maps and ψ_Q is defined by the same formula as ψ. Since B is flat over R, the natural map $B \to B_Q$ is injective. Therefore the natural map

$$B[\![X]\!] \to B_Q[\![X]\!]$$

is injective as well. Since A is flat over R, we have that

$$A \otimes_R B[\![X]\!] \to A \otimes_R B_Q[\![X]\!] \simeq A_Q \otimes_Q B_Q[\![X]\!]$$

is injective as well, meaning that the top horizontal arrow of (4.35) is injective. By the injectivity of (3.10), it follows that ψ_Q is injective as well, therefore so is ψ. $\square$

Lemma 4.34. *The natural map*

$$E^{\mathrm{sep}}_{\Delta_n} \hookrightarrow E^{\mathrm{sep}}_n \otimes_{E_n} E^{\mathrm{sep}\,+}_{\Delta_{n-1}}[\![X_n]\!][X_{\Delta_n}^{-1}]$$

is an embedding.

Proof. For every $1 \leq j \leq n-1$, we have that $E_j^{\mathrm{sep}\,+}$ and $E_{\Delta_{n-1}}$ are flat algebras over E_j^+ because they are torsion free over E_j^+. By Remark 3.11 and a repeated application of Lemma 4.33

$$\begin{aligned}
E_{\Delta_n}^{\mathrm{sep}} &\simeq E_n^{\mathrm{sep}} \otimes_{E_n} \left(E_1^{\mathrm{sep}} \otimes_{E_1} \left(\ldots \otimes_{E_{n-2}} \left(E_{n-1}^{\mathrm{sep}} \otimes_{E_{n-1}} E_{\Delta_n}\right) \ldots\right)\right) \\
&\simeq E_n^{\mathrm{sep}} \otimes_{E_n} \left(E_1^{\mathrm{sep}\,+} \otimes_{E_1^+} \left(\ldots \otimes_{E_{n-2}^+} \left(E_{n-1}^{\mathrm{sep}\,+} \otimes_{E_{n-1}^+} E_{\Delta_n}\right) \ldots\right)\right) \\
&\simeq E_n^{\mathrm{sep}} \otimes_{E_n} \left(E_1^{\mathrm{sep}\,+} \otimes_{E_1^+} \left(\ldots \otimes_{E_{n-2}^+} \left(E_{n-1}^{\mathrm{sep}\,+} \otimes_{E_{n-1}^+} E_{\Delta_{n-1}}^+[\![X_n]\!][X_{\Delta_n}^{-1}]\right) \ldots\right)\right) \\
&\hookrightarrow E_n^{\mathrm{sep}} \otimes_{E_n} \left(E_1^{\mathrm{sep}\,+} \otimes_{E_1^+} \left(\ldots \otimes_{E_{n-2}^+} \left(\left(E_{n-1}^{\mathrm{sep}\,+} \otimes_{E_{n-1}^+} E_{\Delta_{n-1}}^+\right)[\![X_n]\!][X_{\Delta_n}^{-1}]\right) \ldots\right)\right) \\
&\hookrightarrow \ldots \\
&\hookrightarrow E_n^{\mathrm{sep}} \otimes_{E_n} \left(\left(E_1^{\mathrm{sep}\,+} \otimes_{E_1^+} \left(\ldots \otimes_{E_{n-2}^+} \left(E_{n-1}^{\mathrm{sep}\,+} \otimes_{E_{n-1}^+} E_{\Delta_{n-1}}^+\right) \ldots\right)\right)[\![X_n]\!][X_{\Delta_n}^{-1}]\right) \\
&\simeq E_n^{\mathrm{sep}} \otimes_{E_n} E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}],
\end{aligned}$$

as desired. $\square$

Proposition 4.35. *The functor*

$$\mathbb{D} : \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L}) \to \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$$

is essentially surjective.

Proof. Consider the Fontaine functor $\mathbb{V}_{F,n}$ from the category of étale $(\varphi_n, \Gamma_{n,L})$-modules over E_n to the category of continuous representations of $G_{n,L}$ with coefficients in κ_L given by

$$\mathbb{V}_{F,n}(-) \coloneqq (E_n^{\mathrm{sep}} \otimes_{E_n} -)^{\varphi_n = \mathrm{id}}.$$

We will show that $\mathbb{V}_{F,n}(D_n) \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$ and that $\mathbb{D}(\mathbb{V}_{F,n}(D_n)) \simeq D$.

For the first assertion, we first note that by Lemma 4.4 (i) $\mathbb{V}_{F,n}(D_n)$ is a finite dimensional vector space over κ_L, since D_n is an étale $(\varphi_n, \Gamma_{n,L})$-module over E_n by Lemma 4.24. Moreover, by Lemma 4.24 $\mathbb{V}_{F,n}(D_n)$ also admits a linear action from $G_{\Delta_{n-1},L}$. We are left to prove that $\mathbb{V}_{F,n}(D_n)$ satisfies the continuity condition. For that we consider $E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)$.

By Lemma 4.22 (iii) $\mathbb{W}(D_n)$ is a finite free module over $E_n^+ = \kappa_L[\![X_n]\!]$, therefore $E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)$ is a finite free module over $E_n^{\mathrm{sep}\,+}$, thus it has no X_n-torsion and the natural map

$$E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n) \to \left(E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)\right)[X_n^{-1}]$$

is an embedding respecting all the existing actions on both sides. By Lemma 4.4 (ii) applied for $D_{\overline{n},r}^{+*}$, the group $G_{\Delta_{n-1},L}$ acts continuously on

$$\bigcap_{i \in \Delta_{n-1}} \left(E_{\Delta_{n-1}}^{\mathrm{sep}}[X_n]/(X_n^r) \otimes_{\kappa_L[X_n]/(X_n^r)} D_{\overline{n},r}^{+*}\right)^{\varphi_i = \mathrm{id}} \simeq \mathbb{W}(D_n)/X_n^r\mathbb{W}(D_n)$$

for every $r \in \mathbb{N}_{>0}$. It follows that for each $x \in \mathbb{W}(D_n)$ and $r \in \mathbb{N}_{>0}$, there exists an open subgroup U_r of $G_{\Delta_{n-1},L}$ such that

$$\sigma(x) - x \in X_n^r \mathbb{W}(D_n)$$

for all $\sigma \in U_r$. Let $y \in (E_n^{\mathrm{sep}} \otimes_{E_n} D_n)^{\varphi_n = \mathrm{id}}$ and view it as an element of

$$E_n^{\mathrm{sep}} \otimes_{E_n} D_n = \left(E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)\right)[X_n^{-1}].$$

Suppose that $X_n^{r_0} y \in E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)$ for r_0 large enough. Then

$$X_n^{r_0} y = \sum_{j=1}^{k} e_j \otimes d_j$$

for some $e_j \in E_n^{\mathrm{sep}\,+}$ and $d_j \in \mathbb{W}(D_n)$. Consider an open subgroup $U \subseteq G_{\Delta_{n-1},L}$ such that $\sigma(d_j) - d_j \in X_n^{r_0+1}\mathbb{W}(D_n)$ for all $\sigma \in U$, $j \in \{1, \ldots, k\}$. Then

$$X_n^{r_0}(\sigma(y) - y) = \sigma(X_n^{r_0} y) - X_n^{r_0} y = \sum_{j=1}^{k} e_j \otimes (\sigma(d_j) - d_j) \in X_n^{r_0+1}\left(E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)\right)$$

for all $\sigma \in U$. Therefore, since $E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)$ is free over $E_n^{\mathrm{sep}\,+}$, we obtain that

$$\sigma(y) - y \in X_n\left(E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)\right)$$

for all $\sigma \in U$. Applying φ_n on both sides and using that $\varphi_n(\mathbb{W}(D_n)) \subseteq \mathbb{W}(D_n)$, $\varphi_n(E_n^{\mathrm{sep}\,+}) \subseteq E_n^{\mathrm{sep}\,+}$ and $\varphi_n(y) = y$, we obtain that

$$\sigma(y) - y \in X_n^{q^s}\left(E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)\right)$$

for arbitrary $s \in \mathbb{N}_{\geq 1}$. Since $E_n^{\mathrm{sep}\,+}$ is normed with X_n contained in its unique maximal ideal and $E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)$ is a finite free module over $E_n^{\mathrm{sep}\,+}$, it follows that

$$\bigcap_{s \geq 0} X_n^{q^s}\left(E_n^{\mathrm{sep}\,+} \otimes_{E_n^+} \mathbb{W}(D_n)\right) = 0$$

and therefore $\sigma(y) = y$ for all $\sigma \in U$. This means that the action of $G_{\Delta_{n-1},L}$ is continuous on $\mathbb{V}_{F,n}(D_n)$ for the discrete topology on $\mathbb{V}_{F,n}(D_n)$. Also, since D_n is an étale $(\varphi_n, \Gamma_{n,L})$-module over E_n, by Lemma 4.4 (ii), it follows that $G_{n,L}$ acts continuously on $\mathbb{V}_{F,n}(D_n)$, therefore $\mathbb{V}_{F,n}(D_n) \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$.

We now show that $\mathbb{D}(\mathbb{V}_{F,n}(D_n))$ and D are isomorphic. Consider the diagram

$$\begin{tikzcd}
E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_n} D_n \arrow[r] \arrow[d] & E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D \arrow[d] \\
E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!) \otimes_{E_n} D_n \arrow[r] & E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D
\end{tikzcd}$$

where the vertical maps are induced by the inclusion $E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \subseteq E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!)$ and the horizontal ones are the maps sending $e \otimes d$ to ed. The top horizontal map is well defined by Proposition 4.31. The left vertical map is injective because D_n is flat over E_n and the lower horizontal map is bijective by Lemma 4.23, therefore the natural map

$$\begin{aligned} E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_n} D_n &\hookrightarrow E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D \\ e \otimes d &\longmapsto ed \end{aligned} \tag{4.36}$$

must be injective. We then obtain that

$$\begin{aligned} E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}_{F,n}(D_n) &\hookrightarrow \left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_n} E_n^{\mathrm{sep}}\right) \otimes_{\kappa_L} \mathbb{V}_{F,n}(D_n) \\ &\simeq E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_n} (E_n^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}_{F,n}(D_n)) \\ &\simeq E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_n} (E_n^{\mathrm{sep}} \otimes_{E_n} D_n) \\ &\simeq E_n^{\mathrm{sep}} \otimes_{E_n} (E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_n} D_n) \\ &\hookrightarrow E_n^{\mathrm{sep}} \otimes_{E_n} (E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D) \end{aligned}$$

where the embedding in the first line is due to Lemma 4.34, the isomorphism

$$E_n^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}_{F,n}(D_n) \longrightarrow E_n^{\mathrm{sep}} \otimes_{E_n} D_n$$

from the third line holds by Lemma 4.4 (iii) applied for D_n and the embedding in the last line is due to (4.36). Overall, we obtained a natural embedding

$$E_{\Delta_n}^{\mathrm{sep}} \otimes_{\kappa_L} \mathbb{V}_{F,n}(D_n) \longrightarrow E_n^{\mathrm{sep}} \otimes_{E_n} (E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D). \tag{4.37}$$

Take the $H_{\Delta_n,L}$-invariants in both sides of (4.37). For the left hand side we obtain $\mathbb{D}(\mathbb{V}_{F,n}(D_n))$. For the right hand side, compute first the $H_{\Delta_{n-1},L}$-invariants. Since E_n^{sep} is an E_n-vector space on which $H_{\Delta_{n-1},L}$ acts trivially, we have that

$$\begin{aligned} &\left(E_n^{\mathrm{sep}} \otimes_{E_n} \left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D\right)\right)^{H_{\Delta_{n-1},L}} \\ &\simeq E_n^{\mathrm{sep}} \otimes_{E_n} \left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D\right)^{H_{\Delta_{n-1},L}} \end{aligned}$$

by Lemma 3.39. The group $H_{\Delta_{n-1},L}$ acts trivially on D and D is a projective E_{Δ_n}-module, therefore by Lemma 3.39 and Lemma 4.32

$$\begin{aligned} \left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D\right)^{H_{\Delta_{n-1},L}} &\simeq \left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}]\right)^{H_{\Delta_{n-1},L}} \otimes_{E_{\Delta_n}} D \\ &\simeq E_{\Delta_n} \otimes_{E_{\Delta_n}} D \simeq D. \end{aligned}$$

Applying a similar reasoning for the $H_{n,L}$-invariants we obtain that

$$\begin{aligned} \left(E_n^{\mathrm{sep}} \otimes_{E_n} \left(E_{\Delta_{n-1}}^{\mathrm{sep}\,+}[\![X_n]\!][X_{\Delta_n}^{-1}] \otimes_{E_{\Delta_n}} D\right)\right)^{H_{\Delta_n,L}} &\simeq (E_n^{\mathrm{sep}} \otimes_{E_n} D)^{H_{n,L}} \\ &\simeq (E_n^{\mathrm{sep}})^{H_{n,L}} \otimes_{E_n} D \simeq E_n \otimes_{E_n} D \simeq D. \end{aligned}$$

We thus have an embedding $\mathbb{D}(\mathbb{V}_{F,n}(D_n)) \hookrightarrow D$. By Lemma 2.12, we have an isomorphism of E_{Δ_n}-modules

$$D \oplus E_{\Delta_n}^{\oplus k_1} \simeq E_{\Delta_n}^{\oplus k_2}$$

for some $k_1, k_2 \geq 1$. Therefore $\mathrm{rk}_{E_{\Delta_n}}(D) = k_2 - k_1$, where the rank of a finitely generated module over an integral domain is defined to be the size of the largest possible linearly independent subset of the module. We then have an isomorphism of $E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!)$-modules

$$\left(E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D\right) \oplus E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!)^{\oplus k_1} \simeq E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!)^{\oplus k_2}. \tag{4.38}$$

By Lemma 4.22 (iv) and Lemma 4.23 $E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!) \otimes_{E_{\Delta_n}} D$ is a finite free module over $E_{\Delta_{n-1}}^{\mathrm{sep}}(\!(X_n)\!)$ of rank equal the dimension of D_n over E_n. By (4.38) it follows that

$$\dim_{E_n}(D_n) = k_2 - k_1.$$

By Lemma 4.4 (iii) applied for the one variable case, the functor $\mathbb{V}_{F,n}$ is rank-preserving and so is the functor $\mathbb{D}$ by Corollary 4.2 (i), therefore

$$\mathrm{rk}_{E_{\Delta_n}} \mathbb{D}(\mathbb{V}_{F,n}(D_n)) = \dim_{E_n}(D_n) = k_2 - k_1 = \mathrm{rk}_{E_{\Delta_n}} D.$$

Hence $D/\mathbb{D}(\mathbb{V}_{F,n}(D_n))$ is a torsion E_{Δ_n}-module with a $\Gamma_{\Delta_n,L}$-action. The global annihilator of this module is a non-zero $\Gamma_{\Delta_n,L}$-invariant ideal of E_{Δ_n}, which by Lemma 2.9 must equal E_{Δ_n}. Therefore the embedding $\mathbb{D}(\mathbb{V}_{F,n}(D_n)) \hookrightarrow D$ is an isomorphism and we are done. $\square$

Combining the results of Lemma 4.5 and Proposition 4.35, we obtain the theorem we were after in this chapter.

Theorem 4.36. *The functors $\mathbb{V}$ and $\mathbb{D}$ are quasi-inverse equivalences between the categories* $\mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$ *and* $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$.

Chapter 5

Equivalence for p-adic coefficients

In this chapter we show that $\mathbb{D}$ and $\mathbb{T}$ are quasi-inverse functors that realize the equivalence between the categories $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ and $\operatorname{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. In Chapter 4 we already showed the desired equivalence when we restrict our attention to the objects of our categories that are annihilated by π. We explain how we can extend this automatically to show that our categories are equivalent when we restrict our attention to objects that are annihilated by a power of π instead. Then we invoke projective limit arguments that start from the isomorphisms of Proposition 2.20 and Proposition 2.26, while making use of some exactness properties of $\mathbb{D}$ and $\mathbb{T}$ that we establish below.

5.1 The functor $\mathbb{D}$

We begin by studying the properties of $\mathbb{D}$. We first make the simple observation that $\mathbb{D}$ preserves left exactness.

Lemma 5.1. *For an exact sequence*

$$0 \longrightarrow T_0 \longrightarrow T \longrightarrow T_1 \longrightarrow 0$$

in $\operatorname{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$*, the sequence*

$$0 \longrightarrow \mathbb{D}(T_0) \longrightarrow \mathbb{D}(T) \longrightarrow \mathbb{D}(T_1)$$

is exact.

Proof. By Corollary 3.32 (ii), the sequence

$$0 \longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_0 \longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T \longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_1 \longrightarrow 0$$

is exact. Taking the $H_{\Delta_n,L}$-invariants, it follows that

$$0 \longrightarrow \mathbb{D}(T_0) \longrightarrow \mathbb{D}(T) \longrightarrow \mathbb{D}(T_1)$$

is exact. $\square$

The first step towards establishing the needed right-exactness properties of $\mathbb{D}$ is the following direct consequence of Lemma 3.42 (ii).

Lemma 5.2. *For $V \in \mathrm{Rep}_{\kappa_L}(G_{\Delta_n,L})$, we have that*

$$H^1_{\mathrm{cont}}(H_{\Delta_n,L}, E^{\mathrm{sep}}_{\Delta_n} \otimes_{\kappa_L} V) = 0.$$

Proof. Using Proposition 4.1 the basis of $E^{\mathrm{sep}}_{\Delta_n} \otimes_{\kappa_L} V$ over $E^{\mathrm{sep}}_{\Delta_n}$ fixed by $H_{\Delta_n,L}$ induces an isomorphism

$$E^{\mathrm{sep}}_{\Delta_n} \otimes_{\kappa_L} V \longrightarrow (E^{\mathrm{sep}}_{\Delta_n})^{\oplus r}$$

of $H_{\Delta_n,L}$-modules where $r = \dim_{\kappa_L} V$. A 1-cocycle in $H^1_{\mathrm{cont}}(H_{\Delta_n,L}, (E^{\mathrm{sep}}_{\Delta_n})^{\oplus r})$ decomposes into r 1-cocycles in $H^1_{\mathrm{cont}}(H_{\Delta_n,L}, E^{\mathrm{sep}}_{\Delta_n})$. By Lemma 3.42 (ii) the conclusion follows. $\square$

We can now show that $\mathbb{D}$ has the desired properties when applied on torsion representations in $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$.

Proposition 5.3. *Let $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ be a torsion representation, that is a representation with $\pi^m T = 0$ for some $m \in \mathbb{N}_{\geq 1}$.*

(i) For an exact sequence

$$0 \longrightarrow T_0 \longrightarrow T \longrightarrow T_1 \longrightarrow 0$$

in $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, the sequence

$$0 \longrightarrow \mathbb{D}(T_0) \longrightarrow \mathbb{D}(T) \longrightarrow \mathbb{D}(T_1) \longrightarrow 0$$

is exact.

(ii) $\mathbb{D}(T)$ is a finitely generated étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-module over $\mathcal{A}_{\Delta_n}$.

(iii) The map

$$\begin{aligned} \mathrm{ad}_T : \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T) &\longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T \\ a \otimes v &\longmapsto av \end{aligned}$$

induced by the inclusion $\mathbb{D}(T) \subseteq \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T$ is an isomorphism.

Proof. (i) By Lemma 5.1 we know that the sequence

$$0 \longrightarrow \mathbb{D}(T_0) \longrightarrow \mathbb{D}(T) \longrightarrow \mathbb{D}(T_1) \longrightarrow 0$$

is left exact. For a torsion representation $W \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ we know that $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} W$ is a discrete $H_{\Delta_n,L}$-module. Indeed, W is annihilated by π^ℓ for a large enough $\ell \in \mathbb{N}$ and

$$\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} = \mathcal{A}^{\mathrm{ur}}_{\Delta_n} + \pi^\ell \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}},$$

therefore an element in $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} W$ can be written as $\sum_{j\in J} a_j \otimes w_j$ with $a_j \in \mathcal{A}^{\mathrm{ur}}_{\Delta_n}$ and $w_j \in W$. Since the π-adic topology on W is the discrete topology, by the isomorphism

$$H_{\Delta_n,L} \simeq \prod_{i\in\Delta_n} \mathrm{Gal}(\mathcal{B}_i^{\mathrm{ur}}/\mathcal{B}_i),$$

from (3.35) and the formula (3.33) there exists an open normal subgroup of $H_{\Delta_n,L}$ fixing each of the a_j and w_j and thus fixing $\sum_{j\in J} a_j \otimes w_j$. Therefore, using the long exact sequence of continuous cohomology, our sequence can be extended to the exact sequence

$$\begin{aligned}0 \longrightarrow (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_0)^{H_{\Delta_n,L}} &\longrightarrow (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T)^{H_{\Delta_n,L}} \longrightarrow (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_1)^{H_{\Delta_n,L}} \\ &\longrightarrow H^1_{\mathrm{cont}}(H_{\Delta_n,L}, \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_0).\end{aligned}$$

If $\pi T_0 = 0$, then

$$H^1_{\mathrm{cont}}(H_{\Delta_n,L}, \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_0) \simeq H^1_{\mathrm{cont}}(H_{\Delta_n,L}, E^{\mathrm{sep}}_{\Delta_n} \otimes_{\kappa_L} T_0) = 0$$

by Lemma 5.2. In the general case we have that $\pi^m T_0 = 0$ and we use the exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi^{m-1}T_0 & \longrightarrow & T & \longrightarrow & T/\pi^{m-1}T_0 & \longrightarrow & 0 \\
0 & \longrightarrow & \pi^{m-2}T_0/\pi^{m-1}T_0 & \longrightarrow & T/\pi^{m-1}T_0 & \longrightarrow & T/\pi^{m-2}T_0 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & T_0/\pi T_0 & \longrightarrow & T/\pi T_0 & \longrightarrow & T/T_0 \simeq T_1 & \longrightarrow & 0
\end{array}$$

to obtain that the maps

$$\mathbb{D}(T) \twoheadrightarrow \mathbb{D}(T/\pi^{m-1}T_0) \twoheadrightarrow \ldots \twoheadrightarrow \mathbb{D}(T/\pi T_0) \twoheadrightarrow \mathbb{D}(T_1)$$

are surjective.

(ii) Applying $\mathbb{D}(-)$ to the exact sequence

$$0 \longrightarrow \pi T \longrightarrow T \longrightarrow T/\pi T \longrightarrow 0,$$

we obtain by part (i) the exact sequence

$$0 \longrightarrow \mathbb{D}(\pi T) \longrightarrow \mathbb{D}(T) \longrightarrow \mathbb{D}(T/\pi T) \longrightarrow 0. \tag{5.1}$$

Then $\mathbb{D}(T)$ will be a finitely generated $\mathcal{A}_{\Delta_n}$-module provided that the outer terms of (5.1) are. By Corollary 4.2 (i) we already know that $\mathbb{D}(T)$ is such when $\pi T = 0$. Therefore, by induction it follows that $\mathbb{D}(T)$ is finitely generated over $\mathcal{A}_{\Delta_n}$ for arbitrary values of m. To show that $\mathbb{D}(T)$ is étale, for $i \in \Delta_n$ we consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_{\Delta_n} \otimes_{\varphi_i,\mathcal{A}_{\Delta_n}} \mathbb{D}(\pi T) \arrow[r] \arrow[d, "\varphi_i^{\mathrm{lin}}"] & \mathcal{A}_{\Delta_n} \otimes_{\varphi_i,\mathcal{A}_{\Delta_n}} \mathbb{D}(T) \arrow[r] \arrow[d, "\varphi_i^{\mathrm{lin}}"] & \mathcal{A}_{\Delta_n} \otimes_{\varphi_i,\mathcal{A}_{\Delta_n}} \mathbb{D}(T/\pi T) \arrow[r] \arrow[d, "\varphi_i^{\mathrm{lin}}"] & 0 \\
0 \arrow[r] & \mathbb{D}(\pi T) \arrow[r] & \mathbb{D}(T) \arrow[r] & \mathbb{D}(T/\pi T) \arrow[r] & 0
\end{tikzcd}$$

whose vertical maps are the linearized maps. Its rows are exact by Corollary 2.3 and the exactness of (5.1). We know that the right vertical arrow is bijective by Corollary 4.2 (ii). Since πT is annihilated by π^{m-1}, an induction with respect to m shows that the middle vertical arrow is bijective as well.

(iii) From the exact sequence (5.1) we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(\pi T) \arrow[r] \arrow[d, "\mathrm{ad}_{\pi T}"] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T) \arrow[r] \arrow[d, "\mathrm{ad}_T"] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T/\pi T) \arrow[r] \arrow[d, "\mathrm{ad}_{T/\pi T}"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \pi T \arrow[r] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} T \arrow[r] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} T/\pi T \arrow[r] & 0.
\end{tikzcd}$$

whose rows are exact by parts (ii) and (iii) of Corollary 3.32. This shows that ad_T is an isomorphism provided that $\mathrm{ad}_{\pi T}$ and $\mathrm{ad}_{T/\pi T}$ are. Both πT and $T/\pi T$ are annihilated by $\pi^{\min\{1,m-1\}}$, thus an induction with respect to m makes it sufficient to prove our claim for $m = 1$. This is done in Corollary 4.2 (iii). □

We will extend the results of Proposition 5.3 to arbitrary representations in $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. For now we make the following observation.

Lemma 5.4. *For any $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ we have that*

$$\mathbb{D}(T) \simeq \varprojlim_m \mathbb{D}(T/\pi^m T).$$

Proof. We have the isomorphisms

$$\begin{aligned}
\mathbb{D}(T) &= (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T)^{H_{\Delta_n,L}} \\
&\simeq \left(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} \varprojlim_m T/\pi^m T\right)^{H_{\Delta_n,L}} \\
&\simeq \left(\varprojlim_m \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T/\pi^m T\right)^{H_{\Delta_n,L}} \\
&\simeq \varprojlim_m \left(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T/\pi^m T\right)^{H_{\Delta_n,L}} \\
&= \varprojlim_m \mathbb{D}(T/\pi^m T),
\end{aligned}$$

where we used Proposition 2.20 in the second line, Lemma 1.47 (ii) in the third and that taking $H_{\Delta_n,L}$-invariants commutes with limits in the fourth. □

Corollary 5.5. *For $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, the natural map*

$$\mathbb{D}(T) \twoheadrightarrow \mathbb{D}(T/\pi^m T)$$

is surjective for all $m \in \mathbb{N}_{\geq 1}$.

Proof. The maps

$$\mathbb{D}(T/\pi^j T) \longrightarrow \mathbb{D}(T/\pi^i T)$$

are surjective when $j \geq i$ due to the exactness of $\mathbb{D}$ on torsion representations in $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ established in Proposition 5.3 (i). The conclusion follows by Lemma 5.4. □

We can also establish the exactness of $\mathbb{D}$ in one more special situation.

Lemma 5.6. *Let $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ such that T is a free $\mathcal{O}_L$-module. Then*

$$\mathbb{D}(T)/\pi^m\mathbb{D}(T) \simeq \mathbb{D}(T/\pi^m T)$$

for all $m \in \mathbb{N}_{\geq 1}$. In particular, $\mathbb{D}(T)$ is π-adically complete.

Proof. By our assumption about T, the sequence

$$0 \longrightarrow T \xrightarrow{\pi^m} T \longrightarrow T/\pi^m T \longrightarrow 0$$

is exact. Applying $\mathbb{D}(-)$, we obtain the sequence

$$0 \longrightarrow \mathbb{D}(T) \xrightarrow{\pi^m} \mathbb{D}(T) \longrightarrow \mathbb{D}(T/\pi^m T) \longrightarrow 0.$$

It is left exact by Lemma 5.1 and it is right exact by Corollary 5.5, hence

$$\mathbb{D}(T)/\pi^m\mathbb{D}(T) \simeq \mathbb{D}(T/\pi^m T).$$

By Lemma 5.4 it also follows that

$$\mathbb{D}(T) \simeq \varprojlim_m \mathbb{D}(T/\pi^m T) \simeq \varprojlim_m \mathbb{D}(T)/\pi^m \mathbb{D}(T)$$

hence $\mathbb{D}(T)$ is π-adically complete. $\square$

We now show that $\mathbb{D}$ preserves the free objects in our categories.

Lemma 5.7. *For $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ such that T is a free $\mathcal{O}_L$-module, $\mathbb{D}(T)$ is a finite free $\mathcal{A}_{\Delta_n}$-module.*

Proof. Using Lemma 5.6 we know that $\mathbb{D}(T)/\pi\mathbb{D}(T) \simeq \mathbb{D}(T/\pi T)$. By Corollary 4.2 (i), we have that $\mathbb{D}(T/\pi T)$ is a finite free module over $\mathcal{A}_{\Delta_n}/\pi\mathcal{A}_{\Delta_n} \simeq E_{\Delta_n}$. Let $d_1, \dots, d_r \in \mathbb{D}(T)$ be elements whose residue classes form a basis of $\mathbb{D}(T)/\pi\mathbb{D}(T)$ over E_{Δ_n}. We will show that $d_1, \dots, d_r$ is an $\mathcal{A}_{\Delta_n}$-basis of $\mathbb{D}(T)$. Since $\mathbb{D}(T)$ and $\mathcal{A}_{\Delta_n}$ are π-adically complete, it suffices to check that

$$\{d_j \bmod \pi^m \mathbb{D}(T)\}_{1 \leq j \leq r}$$

is a basis of $\mathbb{D}(T)/\pi^m\mathbb{D}(T)$ over $\mathcal{A}_{\Delta_n}/\pi^m\mathcal{A}_{\Delta_n}$ for every $m \in \mathbb{N}_{\geq 1}$. First note that the Nakayama Lemma in the form of Lemma 1.51 shows that $d_1, \dots, d_r$ generate $\mathbb{D}(T)$, hence their residue classes generate $\mathbb{D}(T)/\pi^m\mathbb{D}(T)$ for every $m \in \mathbb{N}_{\geq 1}$. We are left to check their linear independence. Suppose that

$$\sum_{j=1}^{r} a_j d_j \in \pi^m \mathbb{D}(T)$$

for some $a_j \in \mathcal{A}_{\Delta_n}$. Then $a_j = \pi a_j'$ for all $1 \leq j \leq r$ for some $a_j' \in \mathcal{A}_{\Delta_n}$ since the residue classes of $d_1, \dots, d_r$ form a basis of $\mathbb{D}(T)/\pi\mathbb{D}(T)$. The module $\mathbb{D}(T)$ is π-torsion free because $\mathbb{D}(T) \subseteq \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T$ and $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T$ is a free $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-module by our assumption on T, thus π-torsion free by Corollary 3.32 (ii). Therefore

$$\sum_{j=1}^{r} a_j' d_j \in \pi^{m-1} \mathbb{D}(T)$$

and an inductive argument shows that the a_j must be multiples of π^m, hence proving the desired linear independence. $\square$

In Remark 2.19 it was explained that for $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, its subset of torsion elements T^{tor} is an object of $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. Furthermore, T/T^{tor} is a finite free module over $\mathcal{O}_L$. This allows us to prove the exactness of the following sequence.

Lemma 5.8. *For $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, the sequence*

$$0 \longrightarrow \mathbb{D}(T^{\mathrm{tor}}) \longrightarrow \mathbb{D}(T) \longrightarrow \mathbb{D}(T/T^{\mathrm{tor}}) \longrightarrow 0$$

is exact.

Proof. In the exact sequence

$$0 \longrightarrow T^{\mathrm{tor}} \longrightarrow T \longrightarrow T/T^{\mathrm{tor}} \longrightarrow 0,$$

the module $T_1 := T/T^{\mathrm{tor}}$ is free, hence flat over $\mathcal{O}_L$. Therefore

$$0 \longrightarrow T^{\mathrm{tor}}/\pi^m T^{\mathrm{tor}} \longrightarrow T/\pi^m T \longrightarrow T_1/\pi^m T_1 \longrightarrow 0$$

is exact for all $m \geq 1$ and we have a system

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & T^{\mathrm{tor}}/\pi^2 T^{\mathrm{tor}} \arrow[r] \arrow[d] & T/\pi^2 T \arrow[r] \arrow[d] & T_1/\pi^2 T_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T^{\mathrm{tor}}/\pi T^{\mathrm{tor}} \arrow[r] & T/\pi T \arrow[r] & T_1/\pi T_1 \arrow[r] & 0
\end{tikzcd}$$

of compatible short exact sequences whose vertical arrows are the surjective projection maps. Applying $\mathbb{D}$ and using its exactness on torsion representations we obtain a system

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathbb{D}(T^{\mathrm{tor}}/\pi^2 T^{\mathrm{tor}}) \arrow[r] \arrow[d] & \mathbb{D}(T/\pi^2 T) \arrow[r] \arrow[d] & \mathbb{D}(T_1/\pi^2 T_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{D}(T^{\mathrm{tor}}/\pi T^{\mathrm{tor}}) \arrow[r] & \mathbb{D}(T/\pi T) \arrow[r] & \mathbb{D}(T_1/\pi T_1) \arrow[r] & 0
\end{tikzcd}$$

of compatible short exact sequences whose vertical maps are still surjective by Proposition 5.3 (i). Therefore, the left projective system satisfies the Mittag-Leffler condition and thus

$$0 \longrightarrow \varprojlim_m \mathbb{D}(T^{\mathrm{tor}}/\pi^m T^{\mathrm{tor}}) \longrightarrow \varprojlim_m \mathbb{D}(T/\pi^m T) \longrightarrow \varprojlim_m \mathbb{D}(T_1/\pi^m T_1) \longrightarrow 0$$

is exact. By Lemma 5.4 the conclusion follows. □

The following propositions show that the results of Proposition 5.3 hold for arbitrary objects of $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$.

Proposition 5.9. *For $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, $\mathbb{D}(T)$ is a finitely generated étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-module over $\mathcal{A}_{\Delta_n}$. In other words, $\mathbb{D}$ is a well defined functor from $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ to $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$.*

Proof. The module $\mathbb{D}(T^{\mathrm{tor}})$ is finitely generated over $\mathcal{A}_{\Delta_n}$ by Proposition 5.3 (ii), while $\mathbb{D}(T/T^{\mathrm{tor}})$ is finitely generated over $\mathcal{A}_{\Delta_n}$ by Lemma 5.7, since T/T^{tor} is a free $\mathcal{O}_L$-module. The result of Lemma 5.8 implies that $\mathbb{D}(T)$ is finitely generated over $\mathcal{A}_{\Delta_n}$ as well. To deduce that $\mathbb{D}(T)$ is étale, for $i \in \Delta_n$ we use the isomorphisms

$$\begin{aligned}
\mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} \mathbb{D}(T) &\simeq \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} \varprojlim_m \mathbb{D}\left(T/\pi^m T\right) \\
&\simeq \varprojlim_m \mathcal{A}_{\Delta_n} \otimes_{\varphi_i, \mathcal{A}_{\Delta_n}} \mathbb{D}\left(T/\pi^m T\right) \\
&\simeq \varprojlim_m \mathbb{D}\left(T/\pi^m T\right) \\
&\simeq \mathbb{D}(T),
\end{aligned}$$

where in the first line we used Lemma 5.4, in the second Proposition 2.2 (ii), in the third Proposition 5.3 (ii) and in the last we used Lemma 5.4 again. □

Note that for $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, the space $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T)$ is equipped with diagonal Frobenius operators using the existing ones on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ and $\mathbb{D}(T)$, as well as a diagonal action of $G_{\Delta_n,L}$, where $G_{\Delta_n,L}$ acts on $\mathbb{D}(T)$ through its quotient $\Gamma_{\Delta_n,L}$.

Proposition 5.10. *For $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, the map*

$$\begin{aligned}
\mathrm{ad}_T : \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T) &\longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T \\
a \otimes d &\longmapsto ad
\end{aligned}$$

is an isomorphism commuting with the action of $G_{\Delta_n,L}$ and the Frobenius operators φ_i on both sides.

Proof. From the exact sequence

$$0 \longrightarrow T^{\mathrm{tor}} \longrightarrow T \longrightarrow T/T^{\mathrm{tor}} \longrightarrow 0$$

we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T^{\mathrm{tor}}) \arrow[r] \arrow[d, "\mathrm{ad}_{T^{\mathrm{tor}}}"] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T) \arrow[r] \arrow[d, "\mathrm{ad}_T"] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T/T^{\mathrm{tor}}) \arrow[r] \arrow[d, "\mathrm{ad}_{T/T^{\mathrm{tor}}}"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T^{\mathrm{tor}} \arrow[r] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T \arrow[r] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T/T^{\mathrm{tor}} \arrow[r] & 0
\end{tikzcd}$$

and to simplify notation we let $T_1 \coloneqq T/T^{\mathrm{tor}}$. Its upper row is exact by Lemma 5.8 and the flatness of $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ over $\mathcal{A}_{\Delta_n}$ from Corollary 3.32 (iii). The bottom row is exact by the flatness of $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ as an $\mathcal{O}_L$-module from Corollary 3.32 (ii). The map $\mathrm{ad}_{T^{\mathrm{tor}}}$ is bijective by Proposition 5.3 (iii), thus it suffices to show that ad_{T_1} is bijective. We have that

$$\begin{aligned}
\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T_1) &\simeq \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \varprojlim_m \mathbb{D}(T_1)/\pi^m \mathbb{D}(T_1) \\
&\simeq \varprojlim_m \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T_1)/\pi^m \mathbb{D}(T_1) \\
&\simeq \varprojlim_m \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T_1/\pi^m T_1) \\
&\simeq \varprojlim_m \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_1/\pi^m T_1 \\
&\simeq \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} \varprojlim_m T_1/\pi^m T_1 \\
&\simeq \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T_1
\end{aligned}$$

where the first equivalence is by the π-adic completeness of $\mathbb{D}(T_1)$ established in Lemma 5.6, the second by Lemma 1.47 (i) and the fact that $\mathbb{D}(T_1)$ is a finite free $\mathcal{A}_{\Delta_n}$-module by Lemma 5.7, the third by Lemma 5.6, the fourth by the bijectivity of $\mathrm{ad}_{T_1/\pi^m T_1}$ for all $m \in \mathbb{N}_{\geq 1}$ implied by Proposition 5.3 (iii), the fifth by Lemma 1.47 (i) and the last by Proposition 2.20.

The commutativity of the map with the Galois action and the Frobenius operators is established in a similar way as in the proof of Corollary 4.2 (iii). □

Corollary 5.11. *For $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, the module $\mathbb{D}(T)$ is π-adically complete.*

Proof. By Proposition 5.9, $\mathbb{D}(T)$ is a finitely generated module over the π-adically complete Noetherian ring $\mathcal{A}_{\Delta_n}$, therefore it is π-adically complete by [Sta18, Tag 031C]. □

We can also show that the functor $\mathbb{D}$ is exact. The key will be to prove that the claim of Lemma 5.6 holds for arbitrary objects in $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. Before we do that, we make the following observation.

Lemma 5.12. *For $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$,*

$$\mathbb{D}(T)/\pi^m \mathbb{D}(T) \simeq \mathbb{D}(T/\pi^m T)$$

for every $m \in \mathbb{N}_{\geq 1}$.

Proof. Starting with the exact sequence

$$T \xrightarrow{\pi^m} T \longrightarrow T/\pi^m T \longrightarrow 0$$

we still have that

$$\mathbb{D}(T) \xrightarrow{\pi^m} \mathbb{D}(T) \longrightarrow \mathbb{D}(T/\pi^m T) \longrightarrow 0$$

is a complex. Therefore

$$\pi^m \mathbb{D}(T) \subseteq \ker\big(\mathbb{D}(T) \to \mathbb{D}(T/\pi^m T)\big)$$

and we have a natural map $\mathbb{D}(T)/\pi^m\mathbb{D}(T) \to \mathbb{D}(T/\pi^m T)$. We use again the short exact sequence

$$0 \longrightarrow T^{\mathrm{tor}} \longrightarrow T \longrightarrow T/T^{\mathrm{tor}} \longrightarrow 0$$

and we denote $T_1 := T/T^{\mathrm{tor}}$ to simplify notation. The flatness of T_1 over $\mathcal{O}_L$ implies that

$$0 \longrightarrow T^{\mathrm{tor}}/\pi^m T^{\mathrm{tor}} \longrightarrow T/\pi^m T \longrightarrow T_1/\pi^m T_1 \longrightarrow 0$$

is exact, while the exactness of $\mathbb{D}$ on torsion representations from Proposition 5.3 (i) implies that

$$0 \longrightarrow \mathbb{D}(T^{\mathrm{tor}}/\pi^m T^{\mathrm{tor}}) \longrightarrow \mathbb{D}(T/\pi^m T) \longrightarrow \mathbb{D}(T_1/\pi^m T_1) \longrightarrow 0$$

is exact. Therefore we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{D}(T^{\mathrm{tor}})/\pi^m\mathbb{D}(T^{\mathrm{tor}}) \arrow[r] \arrow[d] & \mathbb{D}(T)/\pi^m\mathbb{D}(T) \arrow[r] \arrow[d] & \mathbb{D}(T_1)/\pi^m\mathbb{D}(T_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{D}(T^{\mathrm{tor}}/\pi^m T^{\mathrm{tor}}) \arrow[r] & \mathbb{D}(T/\pi^m T) \arrow[r] & \mathbb{D}(T_1/\pi^m T_1) \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. The right exactness of the upper row follows from Lemma 5.8, while the left exactness is a consequence of the fact that $\mathbb{D}(T/T^{\mathrm{tor}})$ is a free $\mathcal{A}_{\Delta_n}$-module by Lemma 5.7. The left vertical arrow is a bijection because $\mathbb{D}$ is exact on torsion representations, while the right one is a bijection by Lemma 5.6. Therefore the middle arrow is a bijection as well. □

Proposition 5.13. *The functor*

$$\mathbb{D} : \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L}) \to \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$$

is exact.

Proof. Let

$$0 \longrightarrow T_0 \longrightarrow T \longrightarrow T_1 \longrightarrow 0$$

be an exact sequence in $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$. By Lemma 5.1 the functor $\mathbb{D}$ is left exact so we are left to show that $\mathbb{D}(T) \longrightarrow \mathbb{D}(T_1)$ is surjective. Using the exactness of

$$T_0/\pi T_0 \longrightarrow T/\pi T \longrightarrow T_1/\pi T_1 \longrightarrow 0$$

and the exactness of $\mathbb{D}$ on torsion representations, we know that the map

$$\mathbb{D}(T/\pi T) \longrightarrow \mathbb{D}(T_1/\pi T_1)$$

is surjective. By Lemma 5.12, it follows that

$$\mathbb{D}(T)/\pi\mathbb{D}(T) \longrightarrow \mathbb{D}(T_1)/\pi\mathbb{D}(T_1)$$

is surjective. The module $\mathbb{D}(T_1)/\pi\mathbb{D}(T_1) \simeq \mathbb{D}(T_1/\pi T_1)$ is finitely generated over E_{Δ_n} by Corollary 4.2 (i). Let $d_1, \ldots, d_r$ be elements of $\mathbb{D}(T)$ whose images generate $\mathbb{D}(T_1)/\pi\mathbb{D}(T_1)$. The ring $\mathcal{A}_{\Delta_n}$ is π-adically complete and $\bigcap_{m=1}^{\infty} \pi^m \mathbb{D}(T_1) = 0$ because $\mathbb{D}(T_1)$ is π-adically complete by Corollary 5.11, therefore the hypotheses of Nakayama Lemma in the form of Lemma 1.51 are satisfied and thus the images of $d_1, \ldots, d_r$ generate $\mathbb{D}(T_1)$. Therefore

$$\mathbb{D}(T) \longrightarrow \mathbb{D}(T_1)$$

is surjective, as desired. □

5.2 The functor $\mathbb{T}$

We now switch our attention to $\mathbb{T}$ and begin with a lemma about its exactness properties.

Lemma 5.14. *Suppose that*

$$0 \longrightarrow D_0 \longrightarrow D \longrightarrow D_1 \longrightarrow 0$$

is an exact sequence in $\mathrm{Mod}^{ét}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$.

(i) The sequence

$$0 \longrightarrow \mathbb{T}(D_0) \longrightarrow \mathbb{T}(D) \longrightarrow \mathbb{T}(D_1)$$

is exact.

(ii) If $\pi^m D_0 = 0$ *for some* $m \in \mathbb{N}_{\geq 1}$*, then*

$$0 \longrightarrow \mathbb{T}(D_0) \longrightarrow \mathbb{T}(D) \longrightarrow \mathbb{T}(D_1) \longrightarrow 0 \tag{5.2}$$

is exact. In particular if D *is annihilated by a power of* π*, then (5.2) is exact.*

Proof. (i) By the flatness of $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ over $\mathcal{A}_{\Delta_n}$ from Corollary 3.32 (iii), the sequence

$$0 \longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0 \longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D \longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_1 \longrightarrow 0 \tag{5.3}$$

is exact. Taking the fixed points of the φ_i operators will preserve left exactness and thus

$$0 \longrightarrow \mathbb{T}(D_0) \longrightarrow \mathbb{T}(D) \longrightarrow \mathbb{T}(D_1)$$

is exact.

(ii) Using the exact sequence (5.3) we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0 \arrow[r] \arrow[d, "\mathrm{id} - \varphi_n"] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D \arrow[r] \arrow[d, "\mathrm{id} - \varphi_n"] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_1 \arrow[r] \arrow[d, "\mathrm{id} - \varphi_n"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0 \arrow[r] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D \arrow[r] & \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_1 \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. Suppose that $m = 1$, meaning that $\pi D_0 = 0$. Then $D_0 \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, E_{\Delta_n})$ and by Lemma 4.4 (iv), the space $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0 \simeq E^{\mathrm{sep}}_{\Delta_n} \otimes_{E_{\Delta_n}} D_0$ has an $E^{\mathrm{sep}}_{\Delta_n}$-basis $\{d_1, \ldots, d_r\}$ fixed by the Frobenius operators φ_i. By Lemma 3.44 it follows that

$$\mathrm{id} - \varphi_n : \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0 \longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0$$

is surjective with kernel $E^{\mathrm{sep}}_{\Delta_{n-1}} d_1 \oplus \ldots \oplus E^{\mathrm{sep}}_{\Delta_{n-1}} d_r$. Therefore by the snake lemma we have an exact sequence

$$0 \longrightarrow (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0)^{\varphi_n = \mathrm{id}} \longrightarrow (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D)^{\varphi_n = \mathrm{id}} \longrightarrow (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_1)^{\varphi_n = \mathrm{id}} \longrightarrow 0$$

where $(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0)^{\varphi_n = \mathrm{id}} = E^{\mathrm{sep}}_{\Delta_{n-1}} d_1 \oplus \ldots \oplus E^{\mathrm{sep}}_{\Delta_{n-1}} d_r$. We then consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0)^{\varphi_n = \mathrm{id}} \arrow[r] \arrow[d, "\mathrm{id} - \varphi_{n-1}"] & (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D)^{\varphi_n = \mathrm{id}} \arrow[r] \arrow[d, "\mathrm{id} - \varphi_{n-1}"] & (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_1)^{\varphi_n = \mathrm{id}} \arrow[r] \arrow[d, "\mathrm{id} - \varphi_{n-1}"] & 0 \\
0 \arrow[r] & (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_0)^{\varphi_n = \mathrm{id}} \arrow[r] & (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D)^{\varphi_n = \mathrm{id}} \arrow[r] & (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D_1)^{\varphi_n = \mathrm{id}} \arrow[r] & 0
\end{tikzcd}$$

with exact rows and continue to apply our reasoning until we obtain the exact sequence

$$0 \longrightarrow \mathbb{T}(D_0) \longrightarrow \mathbb{T}(D) \longrightarrow \mathbb{T}(D_1) \longrightarrow 0.$$

For arbitrary $m \in \mathbb{N}_{\geq 1}$, we factor

$$0 \longrightarrow D_0 \longrightarrow D \longrightarrow D_1 \longrightarrow 0$$

into the exact sequences

$$0 \longrightarrow \pi^{m-1} D_0 \longrightarrow D \longrightarrow D/\pi^{m-1} D_0 \longrightarrow 0$$

$$0 \longrightarrow \pi^{m-2} D_0/\pi^{m-1} D_0 \longrightarrow D/\pi^{m-1} D_0 \longrightarrow D/\pi^{m-2} D_0 \longrightarrow 0$$

$$\ldots$$

$$0 \longrightarrow D_0/\pi D_0 \longrightarrow D/\pi D_0 \longrightarrow D/D_0 \simeq D_1 \longrightarrow 0$$

and obtain by the above that the maps

$$\mathbb{T}(D) \twoheadrightarrow \mathbb{T}(D/\pi^{m-1} D_0) \twoheadrightarrow \mathbb{T}(D/\pi^{m-2} D_0) \twoheadrightarrow \ldots \twoheadrightarrow \mathbb{T}(D/\pi D_0) \twoheadrightarrow \mathbb{T}(D_1)$$

are surjective. $\square$

Proposition 5.15. *Let $D \in \mathrm{Mod}^{ét}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ such that $\pi^m D = 0$ for some $m \in \mathbb{N}_{\geq 1}$.*

(i) $\mathbb{T}(D)$ is a finitely generated $\mathcal{O}_L$-module.

(ii) The map

$$\begin{aligned} \mathrm{ad}_D : \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \mathbb{T}(D) &\longrightarrow \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D \\ a \otimes v &\longmapsto av \end{aligned}$$

is an isomorphism.

Proof. (i) For $m = 1$ this follows from Lemma 4.4 (ii). For arbitrary $m \in \mathbb{N}_{\geq 1}$, one applies $\mathbb{T}(-)$ to the exact sequence

$$0 \longrightarrow \pi^{m-1} D \xrightarrow{\ \iota\ } D \xrightarrow{\ \mathrm{pr}\ } D/\pi^{m-1} D \longrightarrow 0. \tag{5.4}$$

By Lemma 5.14 (ii) the sequence

$$0 \longrightarrow \mathbb{T}(\pi^{m-1} D) \xrightarrow{\ \mathbb{T}(\iota)\ } \mathbb{T}(D) \xrightarrow{\ \mathbb{T}(\mathrm{pr})\ } \mathbb{T}(D/\pi^{m-1} D) \longrightarrow 0 \tag{5.5}$$

is exact. We use induction to reduce to the case $m = 1$ and the conclusion follows.

(ii) Consider again the exact sequences (5.4) and (5.5). Using the flatness of $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ over $\mathcal{A}_{\Delta_n}$ and over $\mathcal{O}_L$, respectively, one obtains the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \mathbb{T}(\pi^{m-1} D) \arrow[r] \arrow[d, "\mathrm{ad}_{\pi^{m-1} D}"] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \mathbb{T}(D) \arrow[r] \arrow[d, "\mathrm{ad}_D"] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \mathbb{T}(D/\pi^{m-1} D) \arrow[r] \arrow[d, "\mathrm{ad}_{D/\pi^{m-1} D}"] & 0 \\ 0 \arrow[r] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} \pi^{m-1} D \arrow[r] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D \arrow[r] & \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D/\pi^{m-1} D \arrow[r] & 0 \end{tikzcd}$$

whose rows are exact. We can recursively reduce the question to the case $m = 1$, which follows from Lemma 4.4 (iii). $\square$

Lemma 5.16. *Let D be a finitely generated $\mathcal{A}_{\Delta_n}$-module. Then $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D$ is π-adically complete.*

Proof. The ring $\mathcal{A}_{\Delta_n}$ is Noetherian by Lemma 2.1, D is a finitely generated module over $\mathcal{A}_{\Delta_n}$, while $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ is π-adically complete and flat over $\mathcal{A}_{\Delta_n}$ by Corollary 3.32 (iii). Therefore taking $R = \mathcal{A}_{\Delta_n}, I = \pi \mathcal{A}_{\Delta_n}, M = D$ and $N = \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ in Lemma 1.48 gives the desired conclusion. $\square$

Lemma 5.17. *For $D \in \mathrm{Mod}^{ét}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$, we have that*

$$\mathbb{T}(D) \simeq \varprojlim_m \mathbb{T}(D/\pi^m D).$$

Proof. We have that

$$
\begin{aligned}
\mathbb{T}(D) = \bigcap_{i \in \Delta_n} \left(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D \right)^{\varphi_i = \mathrm{id}} &\simeq \bigcap_{i \in \Delta_n} \left(\varprojlim_m (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D)/\pi^m (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D) \right)^{\varphi_i = \mathrm{id}} \\
&\simeq \bigcap_{i \in \Delta_n} \left(\varprojlim_m \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D/\pi^m D \right)^{\varphi_i = \mathrm{id}} \\
&\simeq \varprojlim_m \bigcap_{i \in \Delta_n} \left(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D/\pi^m D \right)^{\varphi_i = \mathrm{id}} \\
&= \varprojlim_m \mathbb{T}(D/\pi^m D)
\end{aligned}
$$

where the first identification is by Lemma 5.16, while the third is by the commutativity of limits with taking the fixed points of the Frobenius operators φ_i. $\square$

Corollary 5.18. *For $D \in \mathrm{Mod}^{ét}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$, the natural map*

$$\mathbb{T}(D) \twoheadrightarrow \mathbb{T}(D/\pi^m D)$$

is surjective for all $m \in \mathbb{N}_{\geq 1}$.

Proof. The maps $\mathbb{T}(D/\pi^j D) \longrightarrow \mathbb{T}(D/\pi^i D)$ are surjective when $j \geq i$ due to the exactness of $\mathbb{T}$ on objects in $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ annihilated by a power of π established in Lemma 5.14 (ii). Therefore by Lemma 5.17 the conclusion follows. $\square$

Lemma 5.19. *For $D \in \mathrm{Mod}^{ét}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$, we have that*

$$\mathbb{T}(D)/\pi^m \mathbb{T}(D) \simeq \mathbb{T}(D/\pi^m D)$$

for all $m \in \mathbb{N}_{\geq 1}$. In particular, $\mathbb{T}(D)$ is π-adically complete.

Proof. We start with the exact sequence

$$D \xrightarrow{\pi^m} D \longrightarrow D/\pi^m D \longrightarrow 0$$

which we split into the short exact sequences

$$0 \longrightarrow D[\pi^m] \longrightarrow D \xrightarrow{\pi^m} \pi^m D \longrightarrow 0$$

and

$$0 \longrightarrow \pi^m D \longrightarrow D \longrightarrow D/\pi^m D \longrightarrow 0.$$

Applying $\mathbb{T}$ we get that

$$0 \longrightarrow \mathbb{T}(D[\pi^m]) \longrightarrow \mathbb{T}(D) \longrightarrow \mathbb{T}(\pi^m D) \longrightarrow 0$$

is exact by Lemma 5.14 (ii) since $D[\pi^m]$ is annihilated by π^m. Also

$$0 \longrightarrow \mathbb{T}(\pi^m D) \longrightarrow \mathbb{T}(D) \longrightarrow \mathbb{T}(D/\pi^m D) \longrightarrow 0$$

is exact by a combination of Lemma 5.14 (i) and Corollary 5.18. Therefore

$$\mathbb{T}(D) \xrightarrow{\pi^m} \mathbb{T}(D) \longrightarrow \mathbb{T}(D/\pi^m D) \longrightarrow 0$$

is exact giving the isomorphism

$$\mathbb{T}(D)/\pi^m \mathbb{T}(D) \simeq \mathbb{T}(D/\pi^m D).$$

The π-adic completeness of $\mathbb{T}(D)$ follows now from Lemma 5.17. $\square$

We can now generalize the result of Proposition 5.15 to arbitrary objects of $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. Note that for $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$, the space $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \mathbb{T}(D)$ is equipped with Frobenius operators using the existing ones on $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$, as well as a diagonal action of $G_{\Delta_n,L}$.

Proposition 5.20. *Let $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$.*

(i) $\mathbb{T}(D)$ is finitely generated over $\mathcal{O}_L$.

(ii) The map

$$\begin{aligned} \mathrm{ad}_D : \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \mathbb{T}(D) &\longrightarrow \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D \\ a \otimes v &\longmapsto av \end{aligned}$$

is an isomorphism commuting with the action of $G_{\Delta_n,L}$ and the Frobenius operators φ_i on both sides.

Proof. (i) By Lemma 5.19 $\mathbb{T}(D)/\pi\mathbb{T}(D) \simeq \mathbb{T}(D/\pi D)$ and the latter is finitely generated over $\mathcal{O}_L/\pi\mathcal{O}_L$ by Lemma 4.4 (ii). $\mathbb{T}(D)$ is a π-adically complete module over the π-adically complete ring $\mathcal{O}_L$, therefore the hypotheses of the Nakayama Lemma in the form of Lemma 1.51 are satisfied and thus $\mathbb{T}(D)$ is finitely generated over $\mathcal{O}_L$.

(ii) We have that

$$\begin{aligned} \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \mathbb{T}(D) &\simeq \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \varprojlim_m \mathbb{T}(D)/\pi^m\mathbb{T}(D) \\ &\simeq \varprojlim_m \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \mathbb{T}(D)/\pi^m\mathbb{T}(D) \\ &\simeq \varprojlim_m \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} \mathbb{T}(D/\pi^m D) \\ &\simeq \varprojlim_m \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D/\pi^m D \\ &\simeq \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D \end{aligned}$$

where the first identification is by the π-adic completeness of $\mathbb{T}(D)$ established in Lemma 5.19, the second is by Lemma 1.47 (ii) due to the finite generation of $\mathbb{T}(D)$

established in the previous part, the third is by Lemma 5.19, the fourth is by the bijectivity of $\mathrm{ad}_{D/\pi^m D}$ established in Proposition 5.15 (ii) and the last is by the π-adic completeness of $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D$ from Lemma 5.16. $\square$

For $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$, by Theorem 2.42 the action of the group $\Gamma_{\Delta_n,L}$ on D is continuous if D is equipped with the weak topology. Therefore the action of $G_{\Delta_n,L}$ on D through its quotient $\Gamma_{\Delta_n,L}$ is continuous as well.

Proposition 5.21. *Let $D \in \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$. The group $G_{\Delta_n,L}$ acts continuously on $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D$ if the latter is equipped with the weak topology from its $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$-module structure.*

Proof. The proof is similar to that of Lemma 3.1.11 in [Sch17]. Let $d_1, \dots, d_r \in D$ be elements that generate D as an $\mathcal{A}_{\Delta_n}$-module. Then the elements $\widetilde{d}_1 := 1 \otimes d_1, \dots, \widetilde{d}_r := 1 \otimes d_r$ generate $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D$ as an $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$-module. By Remark 3.37 (ii) we know that the weak topology of $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ has a fundamental system of open neighbourhoods of zero that is closed under addition and we let $\mathfrak{U}(\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}})$ denote such a system. Note that the weak topology of $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ induces the weak topology on $\mathcal{A}_{\Delta_n}$. Hence

$$\mathfrak{U}(\mathcal{A}_{\Delta_n}) = (U \cap \mathcal{A}_{\Delta_n})_{U \in \mathfrak{U}(\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}})}$$

is a fundamental system of open neighbourhoods of zero in $\mathcal{A}_{\Delta_n}$ for the weak topology that is closed under addition.

By Lemma 1.46 it follows that the sets $\left(\sum_{j=1}^{r} U \widetilde{d}_j\right)_{U \in \mathfrak{U}(\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}})}$ form a fundamental system of open neighbourhoods of zero in $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D$. Also, by Lemma 1.46 applied for $\mathcal{A}_{\Delta_n}$-modules equipped with the weak topology, the sets $\left(\sum_{j=1}^{r} U d_j\right)_{U \in \mathfrak{U}(\mathcal{A}_{\Delta_n})}$ form a fundamental system of open neighbourhoods of zero in D.

Step 1: We will show that for any $\sigma \in G_{\Delta_n,L}, x \in \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D$ and $U \in \mathfrak{U}(\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}})$ there is an open subgroup $H \subseteq G_{\Delta_n,L}$ such that

$$h\sigma(x) - \sigma(x) \subseteq \sum_{j=1}^{r} U \widetilde{d}_j \tag{5.6}$$

for any $h \in H$. Since U is closed under addition, it suffices to consider the pure tensors $x = a \otimes d$ with $a \in \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$ and $d \in D$. We write

$$\sigma(d) = \sum_{j=1}^{r} a_j d_j$$

with $a_j \in \mathcal{A}_{\Delta_n}$. Since multiplication in $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ is continuous and U is closed under addition, we find an open set $U' \in \mathfrak{U}(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}})$ such that

$$U' \cdot U' + \sigma(a)U' + \sum_{j=1}^{r} a_j U' \subseteq U. \tag{5.7}$$

Since $G_{\Delta_n,L}$ acts continuously on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$, we can find an open subgroup $H_1 \subseteq G_{\Delta_n,L}$ such that

$$h\sigma(a) \in \sigma(a) + U' \tag{5.8}$$

for any $h \in H_1$. Since $G_{\Delta_n,L}$ acts continuously on D, we can find an open subgroup $H_2 \subseteq G_{\Delta_n,L}$ such that

$$h\sigma(d) \in \sigma(d) + \sum_{j=1}^{r} (U' \cap \mathcal{A}_{\Delta_n}) d_j \tag{5.9}$$

for all $h \in H_2$. Let $H \coloneqq H_1 \cap H_2$. For $h \in H$, we compute

$$\begin{aligned}
&h\sigma(x) - \sigma(x) = h\sigma(a) \otimes h\sigma(d) - \sigma(a) \otimes \sigma(d) \\
&= \sigma(a) \otimes (h\sigma(d) - \sigma(d)) + (h\sigma(a) - \sigma(a)) \otimes (h\sigma(d) - \sigma(d)) + ((h\sigma(a) - \sigma(a)) \otimes \sigma(d) \\
&\in \sigma(a) \otimes \left(\sum_{j=1}^{r} (U' \cap \mathcal{A}_{\Delta_n})\, d_j \right) + U' \otimes \left(\sum_{i=j}^{r} (U' \cap \mathcal{A}_{\Delta_n})\, d_j \right) + U' \otimes \sigma(d) \\
&\subseteq \sum_{j=1}^{r} (\sigma(a)U' + U'U' + U'a_j)\, \widetilde{d}_j \subseteq \sum_{j=1}^{r} U \widetilde{d}_j
\end{aligned}$$

where the inclusion in the third line holds by (5.8) and (5.9), the penultimate inclusion holds by our formula of $\sigma(d)$ and the last inclusion is due to (5.7).

Step 2: We claim that for any $\sigma \in G_{\Delta_n,L}$ and $U \in \mathfrak{U}(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}})$ there is an open subgroup $H \subseteq G_{\Delta_n,L}$ and a neighbourhood $U' \in \mathfrak{U}(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}})$ such that

$$h\sigma \left(\sum_{j=1}^{r} U' \widetilde{d}_j \right) \subseteq \sum_{j=1}^{r} U \widetilde{d}_j \tag{5.10}$$

for any $h \in H$. This time we write

$$\sigma(d_j) = \sum_{k=1}^{r} a_{jk} d_k$$

with $a_{jk} \in \mathcal{A}_{\Delta_n}$ for $1 \leq j \leq r$. Since multiplication in $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$ is continuous and U is closed under addition, we find an $U'' \in \mathfrak{U}(\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}})$ such that

$$U'' \cdot U'' + \sum_{1 \leq j,k \leq r} a_{jk} U'' \subseteq U. \tag{5.11}$$

The group $G_{\Delta_n,L}$ acts continuously on $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$, therefore we find an open subgroup $H_1 \subseteq G_{\Delta_n,L}$ and an open neighbourhood $U' \subseteq U''$ in $\mathfrak{U}(\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}})$ such that

$$h\sigma\left(U'\right) \subseteq U'' \tag{5.12}$$

for all $h \in H_1$. On the other hand, by the continuity of the action of $G_{\Delta_n,L}$ on D we can find an open subgroup $H_2 \subseteq G_{\Delta_n,L}$ such that

$$h\sigma(d_j) \in \sigma\left(d_j\right) + \sum_{k=1}^{r}\left(U' \cap \mathcal{A}_{\Delta_n}\right) d_k \tag{5.13}$$

for any $1 \leq j \leq r$ and any $h \in H_2$. It follows that for any $h \in H := H_1 \cap H_2$, we have that

$$\begin{aligned} h\sigma\left(\sum_{j=1}^{r} U'\widetilde{d}_j\right) &= \sum_{j=1}^{r} h\sigma\left(U'\right)\left(1 \otimes h\sigma\left(d_j\right)\right) \\ &\subseteq \sum_{j=1}^{r} U''\left(1 \otimes \left(\sigma(d_j) + \sum_{k=1}^{r}\left(U' \cap \mathcal{A}_{\Delta_n}\right) d_k\right)\right) \\ &= \sum_{j=1}^{r}\sum_{k=1}^{r} U''\left(a_{jk} + U'\right)\widetilde{d}_k \\ &\subseteq \sum_{j=1}^{r} U\widetilde{d}_j, \end{aligned}$$

where the inclusion in the second line is by (5.12) and (5.13), the equality in the third line is by our formulas for $\sigma(d_j)$ and the last inclusion is by (5.11) and the fact that $U' \subseteq U''$.

Step 3: Now we let $\sigma \in G_{\Delta_n,L}, x \in \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D$, and $U \in \mathfrak{U}(\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}})$ be arbitrary. We choose an open subgroup $H \subseteq G_{\Delta_n,L}$ and a neighbourhood $U' \in \mathfrak{U}(\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}})$ in such a way that (5.6) and (5.10) hold. Then

$$\begin{aligned} h\sigma\left(x + \sum_{j=1}^{r} U'\widetilde{d}_j\right) &= \sigma(x) + h\sigma(x) - \sigma(x) + h\sigma\left(\sum_{j=1}^{r} U'\widetilde{d}_j\right) \\ &\subseteq \sigma(x) + \sum_{j=1}^{r} U\widetilde{d}_j + \sum_{j=1}^{r} U\widetilde{d}_j \\ &\subseteq \sigma(x) + \sum_{j=1}^{r} U\widetilde{d}_j \end{aligned}$$

for any $h \in H$, which shows the continuity of the $G_{\Delta_n,L}$-action on $\widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{A}_{\Delta_n}} D$. $\square$

In order to show that $\mathbb{T}(D) \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$, we are left to show that the action of $G_{\Delta_n,L}$ on $\mathbb{T}(D)$ is continuous for the π-adic topology on $\mathbb{T}(D)$. We finally achieve this in the following lemma.

Lemma 5.22. *The action of $G_{\Delta_n,L}$ on $\mathbb{T}(D)$ is continuous for the π-adic topology on $\mathbb{T}(D)$. In particular, $\mathbb{T}$ is a well defined functor from $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ to $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$.*

Proof. The diagonal action of $G_{\Delta_n,L}$ on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D$ is continuous by Proposition 5.21, therefore $G_{\Delta_n,L}$ acts continuously on $\mathbb{T}(D)$, if the latter is equipped with the subspace topology of the weak topology of $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D$ viewed as an $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-module.

It suffices to check that this subspace topology coincides with the π-adic topology on $\mathbb{T}(D)$. Note that the composition of the maps

$$\mathbb{T}(D) \xrightarrow{x \mapsto 1\otimes x} \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} \mathbb{T}(D) \xrightarrow{\mathrm{ad}_D} \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D$$

induces the inclusion $\mathbb{T}(D) \subseteq \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D$. Therefore the map

$$\begin{aligned} \mathbb{T}(D) &\longrightarrow \widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} \mathbb{T}(D) \\ x &\longmapsto 1 \otimes x \end{aligned}$$

is an embedding.

By Corollary 1.42, the map ad_D is a homeomorphism if we regard both $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} \mathbb{T}(D)$ and $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D$ as spaces equipped with the weak topology from their $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-module structures, because it is an $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}}$-linear isomorphism by Proposition 5.20 (ii). Because this homeomorphism respects the action of $G_{\Delta_n,L}$, it follows that the diagonal action of $G_{\Delta_n,L}$ on $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} \mathbb{T}(D)$ is continuous for the weak topology on the latter by Proposition 5.21. It suffices to show that $\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} \mathbb{T}(D)$ induces the π-adic topology on $\mathbb{T}(D)$. Because $\mathbb{T}(D)$ is finitely generated over $\mathcal{O}_L$ by Proposition 5.20 (i), using the elementary divisor theorem on $\mathbb{T}(D)$ and Lemma 3.38, the conclusion follows. $\square$

For $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ and $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$ consider the maps

$$\mathrm{adj}_T : T \xrightarrow{x \mapsto 1\otimes x} \bigcap_{i\in\Delta_n} (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} T)^{\varphi_i = \mathrm{id}} \xrightarrow{\mathrm{ad}_T^{-1}} \bigcap_{i\in\Delta_n} (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} \mathbb{D}(T))^{\varphi_i = \mathrm{id}} = \mathbb{T}(\mathbb{D}(T))$$

and

$$\mathrm{adj}_D : D \xrightarrow{d \mapsto 1\otimes d} (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D)^{H_{\Delta_n,L}} \xrightarrow{\mathrm{ad}_D^{-1}} \bigcap_{i\in\Delta_n} (\widehat{\mathcal{A}^{\mathrm{ur}}_{\Delta_n}} \otimes_{\mathcal{O}_L} \mathbb{T}(D))^{H_{\Delta_n,L}} = \mathbb{D}(\mathbb{T}(D)).$$

We are ready to prove the main theorem of our thesis.

Theorem 5.23. *The functors $\mathbb{D}$ and $\mathbb{T}$ are quasi-inverse equivalences of categories between $\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n})$.*

Proof. We show that the maps $\{\mathrm{adj}_T\}_{T\in\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})}$ give a natural isomorphism between the functors $\mathbb{T}\circ\mathbb{D}$ and $\mathrm{id}_{\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})}$. Let $W\in\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ be a torsion representation such that $\pi^m W=0$ for some $m\in\mathbb{N}_{\geq 1}$. Using the exactness of $\mathbb{T}$ and $\mathbb{D}$ for objects annihilated by a power of π, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi W \arrow[r] \arrow[d, "\mathrm{adj}_{\pi W}"] & W \arrow[r] \arrow[d, "\mathrm{adj}_W"] & W/\pi W \arrow[r] \arrow[d, "\mathrm{adj}_{W/\pi W}"] & 0 \\
0 \arrow[r] & \mathbb{T}(\mathbb{D}(\pi W)) \arrow[r] & \mathbb{T}(\mathbb{D}(W)) \arrow[r] & \mathbb{T}(\mathbb{D}(W/\pi W)) \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. The bijectivity of adj_W is reduced to the case $m=1$ which was established in Corollary 4.3.

Therefore, for arbitrary $T\in\mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L})$ the map $\mathrm{adj}_{T/\pi^m T}$ is a bijection for all $m\in\mathbb{N}_{\geq 1}$. We also have that

$$\mathbb{T}(\mathbb{D}(T))\simeq\varprojlim_m\mathbb{T}(\mathbb{D}(T)/\pi^m\mathbb{D}(T))\simeq\varprojlim_m\mathbb{T}(\mathbb{D}(T/\pi^m T))$$

where the first isomorphism follows from Lemma 5.17 and the second from Lemma 5.12. Therefore $\mathrm{adj}_T=\varprojlim_m\mathrm{adj}_{T/\pi^m T}$ and its bijectivity follows from that of each $\mathrm{adj}_{T/\pi^m T}$.

Similarly, we show that the maps $\{\mathrm{adj}_D\}_{D\in\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n},\Gamma_{\Delta_n,L},\mathcal{A}_{\Delta_n})}$ give a natural isomorphism between the functors $\mathbb{D}\circ\mathbb{T}$ and $\mathrm{id}_{\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n},\Gamma_{\Delta_n,L},\mathcal{A}_{\Delta_n})}$. Suppose that $M\in\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n},\Gamma_{\Delta_n,L},\mathcal{A}_{\Delta_n})$ is annihilated by π^m for some $m\in\mathbb{N}_{\geq 1}$. Using the exactness of $\mathbb{T}$ and $\mathbb{D}$ for objects annihilated by a power of π, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi M \arrow[r] \arrow[d, "\mathrm{adj}_{\pi M}"] & M \arrow[r] \arrow[d, "\mathrm{adj}_M"] & M/\pi M \arrow[r] \arrow[d, "\mathrm{adj}_{M/\pi M}"] & 0 \\
0 \arrow[r] & \mathbb{D}(\mathbb{T}(\pi M)) \arrow[r] & \mathbb{D}(\mathbb{T}(M)) \arrow[r] & \mathbb{D}(\mathbb{T}(M/\pi M)) \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. The bijectivity of adj_M is reduced to the case $m=1$ which was established in Lemma 4.5. We also have that

$$\mathbb{D}(\mathbb{T}(D))\simeq\varprojlim_m\mathbb{D}(\mathbb{T}(D)/\pi^m\mathbb{T}(D))\simeq\varprojlim_m\mathbb{D}(\mathbb{T}(D/\pi^m D))$$

where the first isomorphism follows from Lemma 5.4 and the second from Lemma 5.19. Therefore $\mathrm{adj}_D=\varprojlim_m\mathrm{adj}_{D/\pi^m D}$ and its bijectivity follows from that of each $\mathrm{adj}_{D/\pi^m D}$. $\square$

5.3 Equivalence in other coefficients

Let $\mathfrak{A}$ be a commutative $\mathcal{O}_L$-algebra which is finitely generated as an $\mathcal{O}_L$-module. We equip $\mathfrak{A}$ with the π-adic topology, for which $\mathfrak{A}$ becomes a topological ring. We use the theory of Section 1.4 applied for $R = \mathfrak{A}$ and call the linear topology on a finitely generated $\mathfrak{A}$-module the π-adic topology. Since $\mathfrak{A}$ is a finitely generated $\mathcal{O}_L$-module, by Lemma 1.43 this also coincides with the π-adic topology if we regard our $\mathfrak{A}$-module as an $\mathcal{O}_L$-module instead.

Definition 5.24. *A finitely generated linear representation of $G_{\Delta_n,L}$ with coefficients in $\mathfrak{A}$ is called* continuous *if the action of $G_{\Delta_n,L}$ on the underlying $\mathfrak{A}$-module is continuous for its π-adic topology.*

Definition 5.25. *A morphism between two finitely generated continuous representations of $G_{\Delta_n,L}$ with coefficients in $\mathfrak{A}$ is a $G_{\Delta_n,L}$-equivariant $\mathfrak{A}$-linear map. Let $\mathrm{Rep}_{\mathfrak{A}}(G_{\Delta_n,L})$ denote the category of finitely generated continuous representations of $G_{\Delta_n,L}$ with coefficients in $\mathfrak{A}$.*

We also consider the algebra $\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}$. The monoid $\mathcal{T}_{+,\Delta_n,L}$ acts diagonally on $\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}$ using the trivial action on $\mathfrak{A}$ and the existing action of $\mathcal{T}_{+,\Delta_n,L}$ on $\mathcal{A}_{\Delta_n}$.

Definition 5.26. *A $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-module over $\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}$ is a finitely generated module D over $\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}$ equipped with a semilinear action of $\mathcal{T}_{+,\Delta_n,L}$. It is called* étale *if for each $i \in \Delta_n$, the linearized map*

$$\begin{aligned}\varphi_i^{\mathrm{lin}} : (\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}) \otimes_{\varphi_i, \mathfrak{A}\otimes_{\mathcal{O}_L}\mathcal{A}_{\Delta_n}} D &\longrightarrow D \\ (c \otimes a) \otimes d &\longmapsto (c \otimes a)\varphi_i(d)\end{aligned}$$

is an isomorphism.

Definition 5.27. *A morphism between two $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over $\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}$ is an $\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}$-linear map which commutes with the action of $\mathcal{T}_{+,\Delta_n,L}$ on both modules. We denote by $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n})$ the category of étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over $\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}$.*

Lemma 5.28. *(i) We have a forgetful functor*

$$\mathfrak{F}_{\mathcal{R}} : \mathrm{Rep}_{\mathfrak{A}}(G_{\Delta_n,L}) \to \mathrm{Rep}_{\mathcal{O}_L}(G_{\Delta_n,L}).$$

(ii) We have a forgetful functor

$$\mathfrak{F}_{\mathcal{D}} : \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}) \to \mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathcal{A}_{\Delta_n}).$$

Proof. (i) Let $T \in \mathrm{Rep}_{\mathfrak{A}}(G_{\Delta_n,L})$. Since T is a finitely generated $\mathfrak{A}$-module and $\mathfrak{A}$ is finitely generated over $\mathcal{O}_L$, it follows that T is a finitely generated $\mathcal{O}_L$-module.

The π-adic topology of T when regarded as an $\mathfrak{A}$-module coincides with the π-adic topology of T when regarded as an $\mathcal{O}_L$-module, therefore the continuity conditions in both of our categories coincide.

(ii) Let $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A}\otimes_{\mathcal{O}_L}\mathcal{A}_{\Delta_n})$. Since D is a finitely generated $\mathfrak{A}\otimes_{\mathcal{O}_L}\mathcal{A}_{\Delta_n}$-module, it is also finitely generated over $\mathcal{A}_{\Delta_n}$. We are left to show that D is étale when regarded as a module over $\mathcal{A}_{\Delta_n}$. For $i \in \Delta_n$ consider the commutative diagram

$$\begin{tikzcd} \mathcal{A}_{\Delta_n} \otimes_{\varphi_i,\mathcal{A}_{\Delta_n}} D \arrow[r, "\varphi_i^{\mathrm{lin}}"] \arrow[d, "\psi"] & D \arrow[d, equal] \\ (\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}) \otimes_{\varphi_i,\mathfrak{A}\otimes_{\mathcal{O}_L}\mathcal{A}_{\Delta_n}} D \arrow[r, "\varphi_i^{\mathrm{lin}}"] & D \end{tikzcd} \tag{5.14}$$

where the horizontal arrows are the linearization maps and $\psi(a \otimes d) \coloneqq (1 \otimes a) \otimes d$ for $a \in \mathcal{A}_{\Delta_n}$ and $d \in D$. The map ψ is well defined because

$$\begin{aligned} \psi(\varphi_i(b)a \otimes d) &= (1 \otimes \varphi_i(b)a) \otimes d \\ &= (1 \otimes \varphi_i(b))[(1 \otimes a) \otimes d] \\ &= \varphi_i(1 \otimes b)[(1 \otimes a) \otimes d] \\ &= (1 \otimes a) \otimes bd \\ &= \psi(a \otimes bd) \end{aligned}$$

for $b \in \mathcal{A}_{\Delta_n}$. The map

$$\begin{aligned} \rho : (\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}) \otimes_{\varphi_i,\mathfrak{A}\otimes_{\mathcal{O}_L}\mathcal{A}_{\Delta_n}} D &\longrightarrow \mathcal{A}_{\Delta_n} \otimes_{\varphi_i,\mathcal{A}_{\Delta_n}} D \\ (c \otimes a) \otimes d &\longmapsto a \otimes cd \end{aligned}$$

where $c \in \mathfrak{A}$, reverts ψ and is similarly checked to be well defined. Therefore ψ is a bijection. Since D is étale over $\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}$, the bottom horizontal arrow of (5.14) is bijective, therefore so must be the upper horizontal arrow. $\square$

On the ring $\mathfrak{A} \otimes_{\mathcal{O}_L} \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$, we have the Frobenius operators and a $G_{\Delta_n,L}$-action given by the formulas

$$\begin{aligned} \varphi_i(c \otimes a) &:= c \otimes \varphi_i(a), \\ \sigma(c \otimes a) &:= c \otimes \sigma(a), \end{aligned}$$

where $\sigma \in G_{\Delta_n,L}, c \in \mathfrak{A}$, $a \in \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}$, $i \in \Delta_n$ with $\varphi_i(a)$ and $\sigma(a)$ previously defined.

For $T \in \mathrm{Rep}_{\mathfrak{A}}(G_{\Delta_n,L})$, consider the space $\left(\mathfrak{A} \otimes_{\mathcal{O}_L} \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}\right) \otimes_{\mathfrak{A}} T$. The group $G_{\Delta_n,L}$ and the operators φ_i act diagonally on $\left(\mathfrak{A} \otimes_{\mathcal{O}_L} \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}\right) \otimes_{\mathfrak{A}} T$ and let

$$\mathbb{D}_{\mathfrak{A}}(T) \coloneqq \left(\left(\mathfrak{A} \otimes_{\mathcal{O}_L} \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}\right) \otimes_{\mathfrak{A}} T\right)^{H_{\Delta_n,L}}.$$

For $D \in \operatorname{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n})$, consider the space $\left(\mathfrak{A} \otimes_{\mathcal{O}_L} \widehat{\mathcal{A}^{\text{ur}}_{\Delta_n}}\right) \otimes_{\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}} D$. The group $G_{\Delta_n,L}$ and the operators φ_i act diagonally on $\left(\mathfrak{A} \otimes_{\mathcal{O}_L} \widehat{\mathcal{A}^{\text{ur}}_{\Delta_n}}\right) \otimes_{\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}} D$ and let

$$\mathbb{T}_{\mathfrak{A}}(D) \coloneqq \bigcap_{i \in \Delta_n} \left(\left(\mathfrak{A} \otimes_{\mathcal{O}_L} \widehat{\mathcal{A}^{\text{ur}}_{\Delta_n}}\right) \otimes_{\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}} D\right)^{\varphi_i = \text{id}}.$$

Proposition 5.29. *Let $T \in \operatorname{Rep}_{\mathfrak{A}}(G_{\Delta_n,L})$ and $D \in \operatorname{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n})$.*

(i) We have that $\mathbb{T}_{\mathfrak{A}}(D) \in \operatorname{Rep}_{\mathfrak{A}}(G_{\Delta_n,L})$ and a canonical isomorphism

$$\mathfrak{F}_R(\mathbb{T}_{\mathfrak{A}}(D)) \simeq \mathbb{T}(\mathfrak{F}_{\mathcal{D}}(D)).$$

(ii) We have that $\mathbb{D}_{\mathfrak{A}}(T) \in \operatorname{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n})$ and a canonical isomorphism

$$\mathfrak{F}_{\mathcal{D}}(\mathbb{D}_{\mathfrak{A}}(T)) \simeq \mathbb{D}(\mathfrak{F}_{\mathcal{R}}(T)).$$

Proof. (i) The operators φ_i act trivially on $\mathfrak{A}$ and fix the elements of $\mathcal{O}_L$ inside $\widehat{\mathcal{A}^{\text{ur}}_{\Delta_n}}$, therefore $\mathbb{T}_{\mathfrak{A}}(D)$ is an $\mathfrak{A}$-module. Consider the map

$$\begin{aligned} \theta_D : \left(\mathfrak{A} \otimes_{\mathcal{O}_L} \widehat{\mathcal{A}^{\text{ur}}_{\Delta_n}}\right) \otimes_{\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}} D &\longrightarrow \widehat{\mathcal{A}^{\text{ur}}_{\Delta_n}} \otimes_{\mathcal{A}_{\Delta_n}} D \\ (c \otimes a) \otimes d &\longmapsto a \otimes cd \end{aligned}$$

where $c \in \mathfrak{A}$, $a \in \mathcal{A}_{\Delta_n}$ and $d \in D$. It is an isomorphism, since it is reverted by the map sending $a \otimes d$ to $(1 \otimes a) \otimes d$. Also, θ_D respects the actions of $G_{\Delta_n,L}$ and the Frobenius operators on both sides, as can be seen from the equalities

$$\begin{aligned} \theta_D \circ \sigma((c \otimes a) \otimes d) &= \theta_D((c \otimes \sigma(a)) \otimes \sigma(d)) \\ &= \sigma(a) \otimes c\sigma(d) \\ &= \sigma(a) \otimes \sigma(cd) \\ &= \sigma(a \otimes cd) \\ &= \sigma(\theta_D((c \otimes a) \otimes d)) \end{aligned}$$

and

$$\begin{aligned} \theta_D \circ \varphi_i((c \otimes a) \otimes d) &= \theta_D((c \otimes \varphi_i(a)) \otimes \varphi_i(d)) \\ &= \varphi_i(a) \otimes c\varphi_i(d) \\ &= \varphi_i(a) \otimes \varphi_i(cd) \\ &= \varphi_i(a \otimes cd) \\ &= \varphi_i(\theta_D((c \otimes a) \otimes d)) \end{aligned}$$

that hold for all $\sigma \in G_{\Delta_n,L}$ and $i \in \Delta_n$.

Taking the fixed points of the operators φ_i, we obtain an isomorphism of $\mathcal{O}_L$-modules between $\mathbb{T}_{\mathfrak{A}}(D)$ and $\mathbb{T}(\mathfrak{F}_{\mathcal{D}}(D))$ respecting the action of $G_{\Delta_n,L}$ on both sides. Since

$\mathbb{T}(\mathfrak{F}_{\mathcal{D}}(D))$ is finitely generated over $\mathcal{O}_L$ by Proposition 5.15 (i), it follows that $\mathbb{T}_{\mathfrak{A}}(D)$ is finitely generated over $\mathcal{O}_L$, hence finitely generated over $\mathfrak{A}$. By Lemma 5.22, the action of $G_{\Delta_n,L}$ on $\mathbb{T}(\mathfrak{F}_{\mathcal{D}}(D))$ is continuous, hence the action of $G_{\Delta_n,L}$ on $\mathbb{T}_{\mathfrak{A}}(D)$ is continuous for the π-adic topology, when we regard the latter as an $\mathcal{O}_L$-module. Since the π-adic topology of T when regarded as an $\mathfrak{A}$-module coincides with the π-adic topology of T when regarded as an $\mathcal{O}_L$-module, it follows that $\mathbb{T}_{\mathfrak{A}}(D) \in \mathrm{Rep}_{\mathfrak{A}}(G_{\Delta_n,L})$ and $\mathfrak{F}_R(\mathbb{T}_{\mathfrak{A}}(D)) \simeq \mathbb{T}(\mathfrak{F}_{\mathcal{D}}(D))$, as desired.

(ii) Because $H_{\Delta_n,L}$ acts trivially on $\mathfrak{A}$ and $\mathcal{A}_{\Delta_n}$, we know that $\mathbb{D}_{\mathfrak{A}}(T)$ is an $\mathfrak{A}\otimes_{\mathcal{O}_L}\mathcal{A}_{\Delta_n}$-module. Consider the map

$$\begin{aligned} \theta_T : (\mathfrak{A} \otimes_{\mathcal{O}_L} \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}}) \otimes_{\mathfrak{A}} T &\longrightarrow \widehat{\mathcal{A}_{\Delta_n}^{\mathrm{ur}}} \otimes_{\mathcal{O}_L} T \\ (c \otimes a) \otimes x &\longmapsto a \otimes cx \end{aligned}$$

where $c \in \mathfrak{A}$, $a \in \mathcal{A}_{\Delta_n}$ and $x \in T$. It is an isomorphism, since it is reverted by the map sending $a \otimes x$ to $(1 \otimes a) \otimes x$. Also, θ_T respects the actions of $G_{\Delta_n,L}$ and the Frobenius operators on both sides, as can be seen from the equalities

$$\begin{aligned} \theta_T \circ \sigma((c \otimes a) \otimes x)) &= \theta_T((c \otimes \sigma(a)) \otimes \sigma(x)) \\ &= \sigma(a) \otimes c\sigma(x) \\ &= \sigma(a) \otimes \sigma(cx) \\ &= \sigma(a \otimes cx) \\ &= \sigma \circ \theta_T((c \otimes a) \otimes x) \end{aligned}$$

and

$$\begin{aligned} \theta_T \circ \varphi_i((c \otimes a) \otimes x) &= \theta_T((c \otimes \varphi_i(a)) \otimes x) \\ &= \varphi_i(a) \otimes cx \\ &= \varphi_i(a \otimes cx) \\ &= \varphi_i \circ \theta_T((c \otimes a) \otimes x). \end{aligned}$$

Taking the $H_{\Delta_n,L}$-invariants of both sides, we obtain an isomorphism of $\mathcal{A}_{\Delta_n}$-modules between $\mathbb{D}(T)$ and $\mathbb{D}(\mathfrak{F}_{\mathcal{R}}(T))$ respecting the action of $\mathcal{T}_{+,\Delta_n,L}$ on both sides. Since $\mathbb{D}(\mathfrak{F}_{\mathcal{R}}(T))$ is finitely generated over $\mathcal{A}_{\Delta_n}$ by Proposition 5.9, it follows that $\mathbb{D}_{\mathfrak{A}}(T)$ is finitely generated over $\mathcal{A}_{\Delta_n}$, hence finitely generated over $\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}$. Since $\mathbb{D}(\mathfrak{F}_{\mathcal{R}}(T))$ is étale by Proposition 5.9, it follows that the linearized map

$$\begin{aligned} \varphi_i^{\mathrm{lin}} : \mathcal{A}_{\Delta_n} \otimes_{\varphi_i,\mathcal{A}_{\Delta_n}} \mathbb{D}_{\mathfrak{A}}(T) &\longrightarrow \mathbb{D}_{\mathfrak{A}}(T) \\ a \otimes d &\longmapsto a\varphi_i(d) \end{aligned}$$

is bijective. Using the commutative diagram

$$\begin{tikzcd} \mathcal{A}_{\Delta_n} \otimes_{\varphi_i,\mathcal{A}_{\Delta_n}} \mathbb{D}_{\mathfrak{A}}(T) \arrow[r, "\varphi_i^{\mathrm{lin}}"] \arrow[d] & \mathbb{D}_{\mathfrak{A}}(T) \arrow[d, equal] \\ (\mathfrak{A} \otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n}) \otimes_{\varphi_i,\mathfrak{A}\otimes_{\mathcal{O}_L}\mathcal{A}_{\Delta_n}} \mathbb{D}_{\mathfrak{A}}(T) \arrow[r, "\varphi_i^{\mathrm{lin}}"] & \mathbb{D}_{\mathfrak{A}}(T) \end{tikzcd}$$

whose left vertical arrow is the isomorphism sending $a\otimes d$ to $(1\otimes a)\otimes d$, it follows that $\mathbb{D}_{\mathfrak{A}}(T) \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A}\otimes_{\mathcal{O}_L}\mathcal{A}_{\Delta_n})$ and $\mathfrak{F}_{\mathcal{D}}(\mathbb{D}_{\mathfrak{A}}(T)) \simeq \mathbb{D}(\mathfrak{F}_{\mathcal{R}}(T))$, as desired. $\square$

We are ready to show that the categories $\mathrm{Rep}_{\mathfrak{A}}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A}\otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n})$ are equivalent as a direct consequence of Theorem 5.23.

Theorem 5.30. *The functors $\mathbb{T}_{\mathfrak{A}}$ and $\mathbb{D}_{\mathfrak{A}}$ are quasi-inverse equivalences between the categories $\mathrm{Rep}_{\mathfrak{A}}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A}\otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n})$.*

Proof. For $T \in \mathrm{Rep}_{\mathfrak{A}}(G_{\Delta_n,L})$, we have the isomorphisms

$$\begin{aligned}\mathfrak{F}_{\mathcal{R}}(T) &\simeq \mathbb{T}(\mathbb{D}(\mathfrak{F}_{\mathcal{R}}(T)))\\ &\simeq \mathbb{T}(\mathfrak{F}_{\mathcal{D}}(\mathbb{D}_{\mathfrak{A}}(T)))\\ &\simeq \mathfrak{F}_{\mathcal{R}}(\mathbb{T}_{\mathfrak{A}}(\mathbb{D}_{\mathfrak{A}}(T)))\end{aligned}$$

where in the first line we used Theorem 5.23, in the second we used Proposition 5.29 (ii) and in the third we used Proposition 5.29 (i). Because $\mathfrak{F}_{\mathcal{R}}$ is faithfully exact, it follows that

$$T \simeq \mathbb{T}_{\mathfrak{A}}(\mathbb{D}_{\mathfrak{A}}(T)).$$

Similarly, for $D \in \mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \mathfrak{A}\otimes_{\mathcal{O}_L} \mathcal{A}_{\Delta_n})$, we have the isomorphisms

$$\begin{aligned}\mathfrak{F}_{\mathcal{D}}(D) &\simeq \mathbb{D}(\mathbb{T}(\mathfrak{F}_{\mathcal{D}}(D)))\\ &\simeq \mathbb{D}(\mathfrak{F}_{\mathcal{R}}(\mathbb{T}_{\mathfrak{A}}(D)))\\ &\simeq \mathfrak{F}_{\mathcal{D}}(\mathbb{D}_{\mathfrak{A}}(\mathbb{T}_{\mathfrak{A}}(D)))\end{aligned}$$

where in the first line we used Theorem 5.23, in the second we used Proposition 5.29 (i) and in the third we used Proposition 5.29 (ii). Because $\mathfrak{F}_{\mathcal{D}}$ is faithfully exact, it follows that

$$D \simeq \mathbb{D}_{\mathfrak{A}}(\mathbb{T}_{\mathfrak{A}}(D)).$$

$\square$

Considering extended coefficients of characteristic p instead, let κ be a commutative κ_L-algebra which is finitely generated as a κ_L-vector space. Similarly as above, we obtain the category $\mathrm{Rep}_{\kappa}(G_{\Delta_n,L})$ of continuous representations of $G_{\Delta_n,L}$ with coefficients in κ and the category $\mathrm{Mod}^{\text{ét}}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \kappa\otimes_{\kappa_L} E_{\Delta_n})$ of étale $(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L})$-modules over $\kappa\otimes_{\kappa_L} E_{\Delta_n}$. Restricting the result of Theorem 5.30 to objects annihilated by π, we also obtain the following.

Corollary 5.31. *Let κ be a commutative κ_L-algebra which is finitely generated as a κ_L-vector space. The categories $\mathrm{Rep}_{\kappa}(G_{\Delta_n,L})$ and $\mathrm{Mod}^{\acute{e}t}(\varphi_{\Delta_n}, \Gamma_{\Delta_n,L}, \kappa\otimes_{\kappa_L} E_{\Delta_n})$ are equivalent.*

List of Notations

Bibliography

[Ber13] Laurent Berger. Multivariable Lubin-Tate (φ, Γ)-modules and filtered φ-modules. *Math. Res. Lett.*, 20(3):409–428, 2013.

[Bou03] Nicolas Bourbaki. *Algebra II. Chapters 4–7.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2003.

[Bre10] Christophe Breuil. The emerging p-adic Langlands programme. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 203–230. Hindustan Book Agency, New Delhi, 2010.

[CKZ21] Annie Carter, Kiran S. Kedlaya, and Gergely Zábrádi. Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate (φ, γ)-modules, 2021.

[Col10] Pierre Colmez. (φ, Γ)-modules et représentations du mirabolique de $\mathrm{GL}_2(\mathbb{Q}_p)$). In *Représentations p-adiques de groupes p-adiques II: Représentations de $GL_2(\mathbb{Q}_p)$ et (φ, Γ)-modules*, pages 61–153. Paris: Société Mathématique de France, 2010.

[Ell57] Robert Ellis. Locally compact transformation groups. *Duke Math. J.*, 24:119–125, 1957.

[FO10] Jean-Marc Fontaine and Yi Ouyang. *Theory of p-adic Galois representations.* 2010. book in preparation.

[Fon90] Jean-Marc Fontaine. Représentations p-adiques des corps locaux. I. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 249–309. Birkhäuser Boston, Boston, MA, 1990.

[FW79] Jean-Marc Fontaine and Jean-Pierre Wintenberger. Extensions algébrique et corps des normes des extensions APF des corps locaux. *C. R. Acad. Sci. Paris Sér. A-B*, 288(8):A441–A444, 1979.

[GK19] Elmar Grosse-Klönne. A note on multivariable (φ, Γ)-modules. *Res. Number Theory*, 5(1):Art. 6, 9, 2019.

[GK20] Elmar Grosse-Klönne. Supersingular Hecke modules as Galois representations. *Algebra Number Theory*, 14(1):67–118, 2020.

[Haz78] Michiel Hazewinkel. *Formal groups and applications*, volume 78 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[KR09] Mark Kisin and Wei Ren. Galois representations and Lubin-Tate groups. *Doc. Math.*, 14:441–461, 2009.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[Mil20] James Stuart Milne. *Class Field Theory*. `https://www.jmilne.org/math/CourseNotes/CFT.pdf`, 2020.

[Neu99] Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[Sch02] Peter Schneider. *Nonarchimedean functional analysis.* Berlin: Springer, 2002.

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[Sch17] Peter Schneider. *Galois representations and* (φ, Γ)*-modules*, volume 164 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017.

[Ser02] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002.

[Sha69] Rodney Y. Sharp. The Cousin complex for a module over a commutative Noetherian ring. *Math. Z.*, 112:340–356, 1969.

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Yek18] Amnon Yekutieli. Flatness and completion revisited. *Algebr. Represent. Theory*, 21(4):717–736, 2018.

[Záb18a] Gergely Zábrádi. Multivariable (φ, Γ)-modules and smooth o-torsion representations. *Selecta Math. (N.S.)*, 24(2):935–995, 2018.

[Záb18b] Gergely Zábrádi. Multivariate (φ, Γ)-modules and products of Galois groups. *Math. Res. Lett.*, 25(2):687–721, 2018.

Selbstständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 42/2018 am 11.07.2018 angegebenen Hilfsmittel angefertigt habe.

Berlin, April 12, 2021
Gheorghe Pupazan